Understanding Universal Properties

Robert

September 2024

1 Introduction

This is mostly an exercise sheet in understanding what universal properties are doing, without actually introducing the full language and machinery of category theory. Universal properties are probably one of the easiest categorical concepts to understand, and also one of the most powerful concepts.

Universal properties are essentially used to *define* certain mathematical constructions. While the underlying construction of a mathematical object may be nasty, the universal property of that object makes it easy to study, and it turns out that almost every property of said object can be deduced just from its universal property.

The main takeaway from doing some of these problems is the observation that if an object is described by a universal property, then it is automatically unique (up to homeomorphism/isomorphism/whatever). In a sense, there is only "one" universal object when describing something with a universal property.

2 Universal properties in topology

Let us experience universal properties in topology with the following exercises.

Exercise 2.1 (Product topology)**.** Let X, Y be topological spaces. We say that $(P, p_1 : P \to X, p_2 : P \to Y)$ has the **universal property of the product of X and Y** if

For any space Z and continuous maps $f_1 : Z \to X$ and $f_2 : Z \to Y$, there is a *unique* continuous map $g : Z \to P$ that makes the following diagram commute:

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists! g"] \arrow[dr, "f_2"] & \\ X & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

(a) Prove that $(X \times Y, \pi_1 : X \times Y \to X, \pi_2 : X \times Y \to Y)$ where $X \times Y$ is given the product topology and $\pi_1 : X \times Y \to X$ is the map $\pi_1((x, y)) = x$, $\pi_2 : X \times Y \to Y$ is the map $\pi_2((x, y)) = y$ satisfies the universal property of the product of $X \times Y$.

(b) Suppose that $(P, p_1 : P \to X, p_2 : P \to Y)$ also satisfies the universal property of the product of X and Y. Prove that there is a *unique* homeomorphism from P into $X \times Y$, where $X \times Y$ is given the product topology.

Exercise 2.2 (Arbitrary product topology)**.** Let $\{ X_\alpha \}_{\alpha \in \Lambda}$ be a collection of topological spaces. We say that a space P and a collection of continuous maps $\{ p_\alpha \}_{\alpha \in \Lambda}$ where $p_\alpha : P \to X_\alpha$ satisfies the **universal property of the product of X_α's** if

For any space Z and collection of continuos maps $\{ f_\alpha : Z \to X_\alpha \}_{\alpha \in \Lambda}$, there is a *unique* continuous map $g : Z \to P$ such that for *any* α, $g \circ p_\alpha = f_\alpha$.

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists! g"'] \arrow[dr, "f_\alpha"] & \\ P \arrow[r, "p_\alpha"] & X_\alpha \end{tikzcd}$$

The situation is illustrated in the above diagram. Note that it is not actually possible to draw infinitely many commutative diagrams.

(a) Let $X := \prod_{\alpha\in\Lambda} X_\alpha$. Give X the product topology, which is generated by the basis

$$\mathcal{B} = \left\{ U \subseteq X : U = \prod_{\alpha\in\Lambda} U_\alpha,\ U_\alpha \text{ is open in } X_\alpha,\ U_\alpha = X_\alpha \text{ for all but finitely many indices} \alpha \right\}.$$

Feel free to check that $\mathcal{B}$ is indeed a basis for a topology on X. An element of X is a function $\mathbf{x} : \Lambda \to X$ such that $\mathbf{x}(\alpha) \in X_\alpha$ for every α. Define $\pi_\alpha : X \to X_\alpha$ by $\pi_\alpha(\mathbf{x}) = \mathbf{x}(\alpha)$. Prove that $X, \{\, \pi_\alpha : X \to X_\alpha \,\}_{\alpha\in\Lambda}$ satisfies the universal property of the product of X_α's.

(b) Suppose that $P, \{\, p_\alpha : P \to X_\alpha \,\}_{\alpha\in\Lambda}$ also satisfies the universal property of the product of X_α's. Prove that there is a *unique* homeomorphism from P into X, where X is defined as above. *Note: The proof of this is going to be extremely similar to the proof of part (b) above.*

Exercise 2.3 (Quotient topology)**.** Let X be a topological space. Then a topological space Y and a continuous map $q : X \to Y$ is said to satisfy the **universal property of the quotient of** X if

For any space Z and continuous map $f : X \to Y$ which has the property that $q(x) = q(x')$ if and only if $f(x) = f(x')$ (if f has this property, we shall say that f is *constant on the fibers of* q), there is a *unique* continuous map $\tilde{f} : Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Z \\ Y \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

(a) Let X be a topological space and suppose that $\sim$ is an equivalence relation on X. Recall that $X/\sim$ is the set of all equivalence classes of X under $\sim$. Define $p : X \to X/\sim$ by $p(x) = [x]$, which sends a point x to its equivalence class. Topologize $X/\sim$ by declaring that $U \subseteq X/\sim$ is open in $X/\sim$ if and only if $p^{-1}(()U)$ is open in X. Prove that $X/\sim$ and p satisfy the universal property of the quotient of X.

(b) Suppose that Y and $q : X \to Y$ also satisfy the universal property of the quotient of X. Prove that there is a *unique* homeomorphism between $X/\sim$ and Y.

Exercise 2.4 (Coproduct topology)**.** Let X, Y be topological spaces. A topological space C and pair of continuous maps $i_1 : X \to C, i_2 : Y \to C$ is said to satisfy the **universal property of the coproduct of** X **and** Y if

For any space Z and continuous maps $f_1 : Z \to X, f_2 : Z \to Y$, there is a *unique* continuous map $g : C \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f_1"] \arrow[r, "i_1"'] & C \arrow[u, dashed, "\exists! g"'] & Y \arrow[ul, "f_2"'] \arrow[l, "i_2"] \end{tikzcd}$$

(a) Let $X + Y$ be the set defined as

$$X + Y := \{1\} \times X \cup \{2\} \times Y = \{\, (1, x) : x \in X \,\} \cup \{\, (2, y) : y \in Y \,\}.$$

This effectively makes $X + Y$ the disjoint union of X and Y. (We "tag" the elements of X and Y to ensure they stay disjoint, even if X and Y are not disjoint.) Define $c_1 : X \to X + Y$ by $c_1(x) = (1, x)$ and $c_2 : Y \to X + Y$ by $c_2(y) = (2, y)$. Intuitively, think of c_1 as embedding X into $X + Y$, and c_2 as embedding Y into $X + Y$. Give $X + Y$ the topology defined by the following: $U \subseteq X + Y$ is open if and only if both $c_1^{-1}(U)$ is open in X and $c_2^{-1}(U)$ is open in Y. Prove that

(i) The topology on $X + Y$ as defined is a valid topology,

(ii) The maps c_1, c_2 are continuous,

(iii) The space $X + Y$ together with $c_1 : X \to X + Y$ and $c_2 : Y \to X + Y$ satisfy the universal property of the coproduct of X and Y.

(b) Suppose that C is another topological space that satisfies the universal property of the coproduct of X and Y. Prove that there is a *unique* homeomorphism from C into $X + Y$.

Exercise 2.5 (Stone-Cěch Compactification)**.** Let X be a completely regular topological space. (If you do not know what this means, do not worry for now, since it does not matter for the purposes of this exercise). A topological space Y and a continuous injective map $i : X \to Y$ is said to have the **universal property of the Stone-Cěch Compactification of** X if

- Y is compact Hausdorff,
- The image of X, $i(X)$, given the subspace topology from Y, is homeomorphic to X,
- $i(X)$ is dense in Y (meaning that the closure of $i(X)$ is equal to Y), and

For any compact Hausdorff space Z and continuous function $f : X \to Z$, there is a *unique* continuous function $\tilde{f} : Y \to Z$ that makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Z \\ Y \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

Prove that if Y, $i : X \to Y$ and Y', $i' : X \to Y'$ are spaces and continuous injective maps that both satisfy the universal property of the Stone-Cěch compactification of X, there is a *unique* homeomorphism from Y to Y'. Note that the argument for this is pretty much going to be the exact same as the above ones.

3 Universal properties in algebra

Exercise 3.1 (Free vector space)**.** For the purposes of this exercise, we work with a fixed field, call it $\mathbb{F}$. Let S be a set. We say that a vector space V and function $i : S \to V$ satisfies the **universal property of a free vector space over** S if

For any vector space W and function (of sets) $f : S \to W$, there is a *unique* linear map $T : V \to W$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "f"'] & V \arrow[d, dashed, "\exists! T"] \\ & W \end{tikzcd}$$

(a) Suppose V with the function $i : S \to V$, and V' with the function $i' : S \to V'$ both satisfy the universal property of a free vector space over S. Prove that there is a unique isomorphism from V to V'.

(b) (optional) We construct a free vector space on S. If S is empty let $V = \{0\}$. If not, let V be the collection of all functions $f : V \to \mathbb{F}$ such that $f(s) = 0$ for all but finitely many s. We make V into a vector space by defining

$$(f + g)(s) = f(s) + g(s)$$
$$(c\dot{f})(s) = c\dot{f}(s),$$

for all $c \in \mathbb{F}$, $s \in S$ and $f, g \in V$.

(i) Check that V is a vector space over $\mathbb{F}$.

(ii) Let $i : S \to V$ be defined by $i(s) = f_s$, where $f_s : S \to V$ is defined by

$$f_s(x) = \begin{cases} 1 & \text{if } s = x \\ 0 & \text{if } s \neq x. \end{cases}$$

Check that i is an injective map.

(iii) Prove that V together with i satisfies the universal property of a free vector space over S.

4 Universal properties in sets

Let's get a bit more practice with universal properties. We will define some simple categorical notions.

Exercise 4.1 (Initial objects)**.** A set S is called an **initial object** if given any other set Y, there is a *unique* function $f : S \to Y$ from S to Y.

(a) Show that if S and S' are both initial objects, then there is a *unique* bijection from S to S'.

(b) Show that the empty set is an initial object in S.

Exercise 4.2 (Terminal objects)**.** A set S is called a **terminal object** if given any other set Y, there is a *unique* function $f : Y \to S$ from Y into S.

(a) Show that if S and S' are both terminal objects, then there is a *unique* bijection between S and S'.

(b) Show that any set containing only one element is a terminal object.

The following exercise is extremely similar to the one with the product topology. However, it is worth doing to see the similarities and gain more familiarity with universal properties.

Exercise 4.3 (Product of sets)**.** Let X, Y be sets. We say that $(P, p_1 : P \to X, p_2 : P \to Y)$ has the **universal property of the product of X and Y** if

For any set Z and functions $f_1 : Z \to X$ and $f_2 : Z \to Y$, there is a *unique* function $g : Z \to P$ that makes the following diagram commute:

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists! g"] \arrow[dr, "f_2"] & \\ X & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

(a) Prove that $(X \times Y, \pi_1 : X \times Y \to X, \pi_2 : X \times Y \to Y)$ where $X \times Y$ is the cartesian product of X and Y and $\pi_1 : X \times Y \to X$ is the map $\pi_1((x, y)) = x$, $\pi_2 : X \times Y \to Y$ is the map $\pi_2((x, y)) = y$ satisfies the universal property of the product of $X \times Y$.

(b) Suppose that $(P, p_1 : P \to X, p_2 : P \to Y)$ also satisfies the universal property of the product of X and Y. Prove that there is a *unique* bijection from P into $X \times Y$.