Real Analysis Toolbox

Robert

June 2024

Abstract

This is a collection of useful tricks I've collected while learning real analysis. Also, I'm not sure why the abstract is not centered.

Chapter 1

Single variable analysis

To be updated soon

Chapter 2

Measure Theory

2.1 Set theory tricks

Trick 2.1.1. Let X be a set and $A, B \subseteq X$. Then, we have

$$A \cup B = [A \cap B^c] \cup [A^c \cap B] \cup [A \cap B].$$

Example 2.1.2. We shall use this trick to prove that if A, B are μ^*-measurable sets, then $A \cup B$ is also μ^*-measurable. Take $E \subseteq X$, and notice that

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\
&\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap A^c \cap B^c) && \text{By subadditivity} \\
&= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cap B)^c)
\end{aligned}$$

//

Trick 2.1.3. If $E, F \subseteq X$ then $E \setminus F = E \cap F^c$.
Trick 2.1.4. Let $E \subseteq X$. Given any other set $A \subseteq X$, we can write E as the *disjoint* union like

$$E = (E \cap A) \cup (E \cap A^c).$$

Trick 2.1.5 (Disjointify more stuff)**.** Given a collection of sets $\{ E_k \}_{k=1}^{\infty}$, we can define a new family of sets $\{ F_k \}_{k=1}^{\infty}$ by

$$F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i.$$

We have the following useful properties:

- F_k's are pairwise disjoint
- $E_k = \bigcup_{i=1}^{k} F_i$
- $\bigcup_{i=1}^{\infty} E_i = \bigcup_{j=1}^{\infty} F_j$

Trick 2.1.6 (Stacking sets)**.** If $\{ A_j \}_{j=1}^{\infty}$ is a pairwise disjoint collection of sets, we can define a new family $\{ B_k \}_{k=1}^{\infty}$ by

$$B_k = \bigcup_{i=1}^{k} A_i.$$

As usual if the A_j's come from a σ-algebra then B_k stays in the σ-algebra.

We have the following useful properties:

- $B_k \cap A_k = A_k$,
- $B_k \cap A_k^c = B_{k-1}$.

Example 2.1.7. Suppose $\{ A_j \}_{j=1}^{\infty}$ is a pairwise disjoint collection of μ^*-measurable sets. Let $E \subseteq X$, then by defining B_k as in Trick 2.1.6, we have for each n,

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}). \end{aligned}$$

//

2.2 Useful Theorems

Lemma 2.2.1. *Let X be a set and let $\mathcal{A}$ be an algebra on X. Let μ_0 be a premeasure and let μ^* be the outer measure induced by μ_0. Let $\mathcal{A}_\sigma$ be the collection of countable unions of sets in $\mathcal{A}$ and $\mathcal{A}_{\sigma\delta}$ be the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Then, given any $E \subseteq X$ and $\varepsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ such that $A \supseteq E$, and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.*

Corollary 2.2.2. *Given any $E \subseteq X$, there exists $B \in A_{\sigma\delta}$ such that $B \supseteq E$ and $\mu^*(B) = \mu^*(E)$.*

Proof. For each n, let $A_n \in \mathcal{A}_\sigma$ be the set obtained from Lemma 2.2.1 such that $A_n \supseteq E$, and $\mu^*(A_n) \leq \mu^*(E) + 1/n$. Now set $B = \bigcap_{n=1}^{\infty} A_n$. Notice that $B \in \mathcal{A}_{\sigma\delta}$. We also have $B \supseteq E$ as each $A_n \supseteq E$, and thus $\mu^*(B) \geq \mu^*(E)$. Now, for every n,

$$\mu^*(B) \leq \mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$$

Letting $n \to \infty$ (taking limits on both sides), we get $\mu^*(B) \leq \mu^*(E)$. □