1 Properties of Lebesgue integrals

Theorem restated from last week.

Theorem 1.1 (Page 75-77). Let $f, g : E \to \mathbb{R}$ be measurable, bounded and $m(E) < \infty$. If f, g are Lebesgue integrable on E then

1. (Linearity) For any $\alpha, \beta \in \mathbb{R}$, $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$
2. (Monotonicity) If $f \leq g$, then $\int_E f \leq \int_E g$.
3. If we had $l, u \in \mathbb{R}$ such that $l \leq f(x) \leq u$ (for all $x \in E$) then $l \cdot m(E) \leq \int_E f \leq u \cdot m(E)$
4. If $E_1, E_2 \in \mathcal{M}$ such that $E_1, E_2 \subseteq E$, $E_1 \cap E_2 = \varnothing$, then $\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f$.

Proof of Linearity. We first show that $\int_E \alpha f = \alpha \int_E f$. This is trivial if $\alpha = 0$. Suppose first that $\alpha > 0$. Recall that $\inf(cA) = c \inf A$, whenever $c > 0$. So it follows immediately by definition of Lebesgue integral and linearity of integral of simple functions. When $c < 0$ we have that $\inf cA = c \sup A$, so use that instead.

Now we show additivity. Observe that if $\phi_1 \leq f$ and $\phi_2 \leq g$, then $\phi_1 + \phi_2 \leq f + g$. So use the definition where you have the sup of the simple functions that are less than $f + g$ and you're basically done. ☐

Remark 1.2. This is a painful way to do this. If we do it the way that Folland does it it is less painful. (Using the Monotone Convergence Theorem instead) See Theorem 2.15 for the additivity of the integral. To generalize this to functions that are real valued, simply decompose them into positive and negative parts.