1 The Lebesgue Integral of a function

Definition 1.1 (Page 73). Let $f : E \to \mathbb{R}$ be a bounded measurable function, $m(E) < \infty$. The **Lebesgue integral of f over E** is defined to be

$$\int_E f = \sup\left\{\int_E \phi | \phi \text{ simple}, \phi \leq f\right\} \qquad \text{Lower lebesgue integral}$$
$$= \inf\left\{\int_E \psi | \psi \text{ simple}, \psi \geq f\right\} \qquad \text{Upper lebesgue integral}$$

Note that the upper and the lower Lebesgue integrals have to agree for this value to be defined.

It is important to note that function inequality is defined pointwise. For $\phi \leq f$ it means for all points x, $\phi(x) \leq f(x)$.

Remark 1.2. See Folland page 50 for an alternative and equivalent definition. Note that this is only for nonnegative extended-real valued functions. For extending this definition to (extended) real valued function, see page 53

Remark 1.3. Obviously f doesn't even have to be bounded and $m(E) < \infty$ is unnecessary too. We can just have an integral which evaluates to infinity.

2 Riemann integrable functions are Lebesgue integrable too

Theorem 2.1 (Page 73). If $f : [a, b] \to \mathbb{R}$ be measurable and bounded. If f is Darboux integrable, then f is Lebesgue integrable, and $\int_a^b f(x)\mathrm{d}x = \int_{[a,b]} f$

Proof. Denote $I = [a, b]$. Let f be as in the hypothesis and suppose f is Darboux integrable. Denote $\mathcal{P}$ to be the set of all possible partitions of I. Observe that

$$\int_a^b f(x)\mathrm{d}x = \inf\{U(f, P) : P \in \mathcal{P}\}$$
$$= \inf_{f \leq \phi}\left\{\int_I \phi \mid \phi \text{ is a step function}\right\}$$

Since

$$\{\ \phi : \phi \text{ simple}\ \} \supseteq \{\ \phi : \phi \text{ step}\ \}$$

it must be that

$$\left\{\int_I \phi \,\middle|\, \phi \text{ simple}\right\} \supseteq \left\{\int_I \phi \,\middle|\, \phi \text{ step}\right\}$$

So we know that

$$\inf_{\phi \geq f}\left\{\int_I \phi \,\middle|\, \phi \text{ simple}\right\} \leq \inf_{\phi \geq f}\left\{\int_I \phi \,\middle|\, \phi \text{ step}\right\}$$

Analogously,

$$\int_a^b f(x)\mathrm{d}x = \sup\{\, L(f,P) : P \in \mathcal{P}\,\}$$
$$= \sup_{\psi \leq f} \left\{ \int_I \psi : \psi \text{ is a step function} \right\}$$

And for the same reasons as above

$$\sup_{\psi \leq f} \left\{ \int_I \phi \;\middle|\; \phi \text{ simple} \right\} \geq \sup_{\psi \leq f} \left\{ \int_I \psi : \psi \text{ step} \right\}$$

Now chain inequalities until you get the desired result. □

Remark 2.2. It was not justified in class why

$$\inf\{U(f,P) : P \in \mathcal{P}\} = \inf_{f \leq \phi} \left\{ \int_I \phi \mid \phi \text{ is a step function} \right\}$$

However this is easy to see just by the definition of $U(f,P)$.

Alternative proof sketch: Recall that in our definition of (upper) Darboux integral we had things that looked like $\sum_{i=1}^n M_i(x_i - x_{i-1})$. Also recall that $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$. m_i is defined analogously with infs instead of sups. We can define a bunch of simple functions $\sum_{i=1}^n M_i \chi_{(x_{i-1},x_i]}$ and notice that the Lebesgue integrals of these guys converge to the upper/lower darboux integral.

Remark 2.3. The condition that f be measurable (or even bounded) is unnecessary. The reason why we supposed it is because proving that it is actually measurable requires the use of the dominated convergence theorem. However all that is technically needed is that f is Darboux integrable. See Theorem 2.28 in Folland.

Theorem 2.4 (Page 75-77)**.** Let $f, g : E \to \mathbb{R}$ be measurable, bounded and $m(E) < \infty$. If f, g are Lebesgue integrable on E then

1. (Linearity) For any $\alpha, \beta \in \mathbb{R}$, $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$
2. (Monotonicity) If $f \leq g$, then $\int_E f \leq \int_E g$.
3. If we had $l, u \in \mathbb{R}$ such that $l \leq f(x) \leq u$ (for all $x \in E$) then $l \cdot m(E) \leq \int_E f \leq u \cdot m(E)$
4. If $E_1, E_2 \in \mathcal{M}$ such that $E_1, E_2 \subseteq E$, $E_1 \cap E_2 = \varnothing$, then $\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f$.

Proof. Not done in class. See proposition 2.13 in Folland. Could be a nice exercise though. □

Remark 2.5. Property 4 is a weaker version of the observation that the map $F \mapsto \int_F f$ is a measure. See exercise 14 of chapter 2 in Folland.

Corollary 2.6. *If* $f : E \to \mathbb{R}$ *is measurable, bounded and* $m(E) < \infty$ *then we have that* $\left|\int_E f\right| \leq \int_E |f|$.

Proof. Trivial and left as an exercise for the reader. See appendix for sketch. □

Remark 2.7. See proposition 2.13 in Folland for alternative proof.

A Proof of some stuff

Proof of Corollary 2.6

Proof. Apply property 2 and the fact that $f \leq |f|$. Obviously $-f \leq |f|$, so apply property 2. Also, $f \leq |f|$ so apply property 2 again. So we have the inequalities

$$\int_E -f \leq \int_E |f|$$
$$\int_E f \leq \int_E |f|$$

Now just combine these and you're basically done. □