List of Theorems

1 Recall

Recall from last lecture
Lemma 1.1 (Fatou's Lemma)**.** *Let $E \in \mathcal{M}$, $m(E) \leq \infty$. Let $\{ f_n \}$ be a sequence of measurable non-negative functions with domain E. If $f_n \to f$ pointwise a.e. on E, then $\int_E f \leq \liminf \int_E f_n$*

Proof. annoying □

2 Monotone Convergence Theorem

Theorem 2.1 (Monotone Convergence Theorem)**.** Let $E \in \mathcal{M}$ and $m(E) \leq \infty$. Let $\{ f_n \}$ be a sequence of extended real-valued measurable functions with domain E, such that $0 \leq f_1(x) \leq \cdots \leq f_n(x) \leq f_{n+1}(x) \leq \cdots$ on E. If $f_n \to f$ pointwise a.e. on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Remark 2.2. Some books require pointwise (not a.e.), some books keep it only real valued. The proof is similar for any of these. For instance, Folland would require that it converges pointwise, not just a.e. This is because Folland has propositions that let you deal with a.e. in great generality.

The proof will begin by applying Fatou's Lemma immediately. This gives us $\liminf \int_E f_n \geq \int_E f$. What we need to do is to get $\int_E f \geq \limsup \int_E f_n$. Recall that if x_n is a sequence and we have $\limsup x_n \leq \lim x_n \leq \liminf x_n$ then we have $\limsup x_n = \liminf x_n = \lim x_n$.

Proof. By Fatou's Lemma (Lemma 1.1), we already have $\int_E \leq \liminf \int_E f_n$. Now, since $f_n \leq f_{n+1}$, we have $f_n \leq f$ a.e., so we have that $\int_E f_n \leq \int_E f$ (follows by basic integral properties). (Another way to see this is that $\{ \int_E f_n \}$ is nothing but a sequence of numbers, so it is immediate.) Now this means that $\limsup \int_E f_n \leq \limsup \int_E f = \int_E f$. So we are done. □

Remark 2.3. You can prove Fatou's Lemma (Lemma 1.1) with MCT (Theorem 2.1). See Folland, Lemma 2.18. In fact Fatou's Lemma and MCT are equivalent.
Exercise 2.4. Assume the Monotone Convergence Theorem and prove Fatou's Lemma. *Hint: No hints, this is C level course.*

For the answer to this exercise, see Folland Lemma 2.18.
Example 2.5. Let $\alpha \in \mathbb{R}$ and let $f : [0,1] \to [0,\infty]$, given by

$$f(x) = \begin{cases} x^{-\alpha} & \text{if } x \in [0,1] \\ \infty & \text{if } x = 0 \end{cases}$$

Notice that we can define a sequence of functions

$$f_n(x) = \begin{cases} x^{-\alpha} & \text{if } x \in [1/n,1] \\ n^{\alpha} & \text{if } x \in [0,1/n) \end{cases}$$

Of course each f_n is measurable. It is easy to see that $\{ f_n \} \to f$ on $[0,1]$, and it is increasing. Thus we apply the monotone convergence theorem to conclude that $\lim \int f_n = \int f$. (Note that the domain we are integrating over has been omitted, but it is $[0,1]$. I did not put it in the subscript to make it cleaner.) //

An immediate application of the monotone convergence theorem is to show that the integral is countably additive, see Folland Theorem 2.15.

3 Dominated Convergence Theorem

The Lebesgue Dominated Convergence Theorem is easily the most powerful convergence theorem. The conditions on the sequence of functions are looser than other convergence theorems:

1. Domain we are integrating on need not be bounded (unlike the bounded convergence theorem)
2. Sequence of functions do not have to be increasing.
3. The pointwise limit function need not be assumed to be integrable.

Theorem 3.1 (Lebesgue's Dominated Convergence Theorem)**.** Let $E \in \mathcal{M}$, and $m(E) \leq \infty$. Let $\{ f_n \}$ be a sequence of real-valued measurable functions with domain E. If $f_n \to f$ a.e. on E and there is a g, nonnegative, integrable on E, such that $|f_n(x)| \leq g(x)$ a.e., then f is integrable on E, and $\lim \int_E f_n = \int_E f$.

Keep in mind that integrable means $\int |g| < \infty$. Since $g = |g|$ as g is nonnegative, $\int g = \int |g|$.

For the proof, we shall produce 2 sequences of nonnegative functions: $g - f_n$ and $g + f_n$[1] We take their integrals, and apply Fatou's Lemma. A neat trick to recall is that given a sequence (x_n), we have $\liminf -x_n = -\limsup x_n$. You will see this trick happening in Equation (2). This will leave us with $\liminf f_n \geq \int f \geq \limsup f_n$ which will prove the result.

Proof. Clearly f is measurable. To see that f is integrable, we can see that $\lim |f_n| = \lim |f| \leq g$, so $\int |f| \leq \int g < \infty$. Now consider the sequences of functions, $(g - f_n)$ and $(g + f_n)$. Both of these sequences are measurable and nonnegative. (Proving that this is true is left as an exercise. Intuitively, we can see that if f_n is being negative then $g + f_n$ is still bigger than 0 since g is bigger than the size of the negativity of f_n. Similar idea for $g + f_n$). It is easy to see that $g + f_n \to g + f$, and $g - f_n \to g - f$. Now, by Fatou's lemma, linearity of the integral, and the fact that $\int g$ is a constant, we have

$$\int g + \int f = \int g + f \leq \liminf \int (g + f_n) = \int g + \liminf \int f_n \tag{1}$$

And again by Fatou's Lemma, we have:

$$\int g - f \leq \liminf \int g - f_n = \liminf \int g + \liminf \left(- \int f_n \right) = \int g - \limsup \int f_n \tag{2}$$

So from Equation (1), getting rid of the constant $\int g$, we are left with $\int f \leq \liminf \int f_n$. From Equation (2), we get that $-\int f \leq -\limsup \int f_n$[2]. Combining these we have $\liminf \int f_n \geq \int f \geq \limsup \int f_n$, so we are done. ☐

As a note to Equation (1), it would seem like we did a bunch of unnecessary work, and that we could have immediately used Fatou's lemma on f_n and f. However, Fatou's lemma is only valid if the f_n's are nonnegative, which they are not assumed to be.

Also, notice that we used the fact that $\int g < \infty$ to subtract it off.

Remark 3.2. If we let the measure be the counting measure on $\mathbb{N}$ we can actually get a useful double summation property out. See Folland Exercise 2.22

Remark 3.3. It is possible to generalize this theorem. We did not really heavily use the fact that $|f_n| \leq g$. What we can do is suppose that we had a sequence of nonnegative g_n's converging pointwise to g a.e., such that $\int g_n \to \int g$ such that $|f|_n \leq g_n$

[1] Note that in class, we took $|g| - f_n$ and $|g| + f_n$ instead. However since $g = |g|$ since g is nonnegative, I decided not to

[2] Since none of these integrals are infinity, we can pull the negative trick. (Again, $\int f$ is finite! That's why we spent time at the start of the proof proving that f is integrable.)