Week 11

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List of Definitions

List of Propositions

1 Separation axioms

Here, the word separation shall mean that we can put disjoint open neighborhoods around things.

Definition 1.1 (T_1 space)**.** A space X is T_1 if points are closed.

Definition 1.2 (T_2 space)**.** A space is T_2 if it is Hausdorff. Recall that a space is Hausdorff if you can separate points with disjoint closed sets.

Definition 1.3 (T_3 space)**.** A space is **regular** or T_3 if it is T_1 and given a point x and a closed set C such that $x \notin C$, there is a neighborhood U of x and $V \supseteq C$ such that U, V are disjoint.

In other words, a space is T_3 if we can separate points from closed sets.

Definition 1.4 ($T_3.5$ space)**.** A space is **completely regular** if it is T_1 and given a point x and a closed set C not containing x, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f[C] = \{ 1 \}$.

Definition 1.5 (T_4 space)**.** A space is **normal** if it is T_1 and given disjoint closed sets C, D, there exists disjoint open sets U, V such that $C \subseteq U$ and $D \subseteq V$.

Clearly normal implies regular implies Hausdorff. A completely regular space is also regular. Normal also implies completely regular, but this is nontrivial (Lemma 1.15).

Proposition 1.6 (Completely regular implies regular)**.** A completely regular space is regular.

Proof. Suppose X is completely regular. Let x be a point and A a closed set not containing x. Then, there is a function $f : X \to [0,1]$ such that $f(x) = 0$ and f is 1 on A. Now preimage some disjoint open neighborhoods of 0 and 1. □

The implications are not reversible.

Example 1.7 (T_1 but not Hausdorff)**.** Let $X = \mathbb{R}$ with the cofinite topology. Then X is clearly T_1 since singletons are finite and are thus closed. It is easily seen to be not Hausdorff. //

Example 1.8 (Hausdorff but not regular)**.** Let $X = \mathbb{R}_K$ be the K-topology on the reals. However, it is not regular since we cannot separate the set K from the point 0. //

Lemma 1.9 (Equivalent condition to regularity)**.** Let X be a topological space. Then X is regular if and only if for every $x \in X$ and neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subseteq U$.

Proof. For the forward direction, apply the definition of regularity to the closed set $X \setminus U$. For the converse direction, if C is a closed set not containing x, then $X \setminus C$ is a neighborhood of x. □

Lemma 1.10 (Equivalent condition to normality)**.** Let X be a topological space. Then X is normal if and only if given a closed set $C \subseteq X$ and an open set $U \supseteq C$, there exists an open set V such that $C \subseteq V$ and $\overline{V} \subseteq U$.

Proof. Same idea as Lemma 1.9. □

Proposition 1.11 (Regularity of subspace and products).
1. A subspace of a (completely) regular space is (completely) regular.
2. Products of (completely) regular space is (completely) regular.

Proof.
1. Let X be regular, and $Y \subseteq X$. Let $y \in Y$ and $A \subseteq Y$ be closed with $y \notin A$. Then there is some $C \subseteq X$ closed such that $A = C \cap Y$. Clearly $x \notin C$ so we may find disjoint U, V containing x and C respectively. Then $U \cap Y$, $V \cap Y$ are the neighborhoods as desired. If X is completely regular instead, let $f : X \to [0,1]$ be a function that separates y and C. The restriction of f to Y is still continuous and has the desired property.
2. Let X_α, $\alpha \in \Lambda$ be a collection of regular spaces. Set $X = \prod_{\alpha\in\Lambda} X_\alpha$. Given a point $\mathbf{x} = \langle x_\alpha : \alpha \in \Lambda\rangle \in X$, let $U = \prod_{\alpha\in\Lambda} U_\alpha$ be an open neighborhood of $\mathbf{x}$, so that $U_\alpha = X_\alpha$ for all but finitely many α. For those U_α which are not equal to X_α, we may find V_α such that $x_\alpha \in V_\alpha$ and $\overline{V_\alpha} \subseteq U_\alpha$ (by Lemma 1.9). For those U_α which are equal to X_α, set V_α to be X_α. Then $V := \prod_{\alpha\in\Lambda} V_\alpha$ has closure contained in U, since $\overline{\prod_{\alpha\in\Lambda} V_\alpha} = \prod_{\alpha\in\Lambda} \overline{V_\alpha}$.

   Now suppose the X_α's are now completely regularly. Let A be closed in X and not containing x. Let $U \subseteq X$ be a basic open neighborhood of $\mathbf{x}$ that is disjoint from A. (This uses regularity of X_α, but finitely many of them). Now, U is the product of open sets U_α in X_α's. Then there are U_α's which are not X_α's, finitely many of them, say $\alpha_1, \dots, \alpha_n$. For each α_i, let $f_i : X_{\alpha_i} \to [0,1]$ be a continuous function such that $f_i(x_{\alpha_i}) = 0$, and f is 1 on the set $X_{\alpha_i} \setminus U_{\alpha_i}$. Define g_i to be $f_i \circ \pi_{\alpha_i}$ and let $g(x) = \prod_{i\le n} g_i(x)$. Then g is the desired function.

□

A similar proposition is not true for normal spaces.
Example 1.12 (Product of normal may not be normal)**.** Let $X = \mathbb{R}_l$ be the Sorgenfrey line. Then X is normal. Let A, B be disjoint closed sets. For $a \in A$, let $[a, x_a) \subseteq X$ be disjoint from B. This is possible because $X \setminus A$ is open. We repeat the same trick for each $b \in B$, letting $[b, x_b) \subseteq X$ be disjoint from A. Now, $A \subseteq \bigcup_{a\in A}[a, x_a)$ and $B \subseteq \bigcup_{b\in B}[b, x_b)$. We claim that those unions are disjoint. Otherwise, let $a \in A, b \in B$ such that $[a, x_a) \cap [b, x_b) \neq \varnothing$. WLOG let us assume $a < b$, so this means $x_a > b$. But then $b \in [a, x_a)$, oops.

However, $X \times X$ is not normal. Let L be the line consisting of the points $\langle x, -x\rangle$. This set is closed[1], and it is discrete, so every subset of L is closed in $X \times X$. For all $A \subseteq L$, we have disjoint open sets U_A, V_A of $X \times X$, such that $A \subseteq U_A$, $L \setminus A \subseteq V_A$. Let us now define a function $f : \mathcal{P}(L) \to \mathcal{P}(\mathbb{Q}^2)$ as follows:

$$
\begin{aligned}
f(\varnothing) &= \varnothing, \\
f(L) &= \mathbb{Q}^2, \\
f(A) &= U_A \cap \mathbb{Q}^2 && \text{if } \varnothing \subset A \subset L.
\end{aligned}
$$

We claim that f is injective. To see this, let A be a subset of L. If A is nonempty, then U_A is nonempty. As $\mathbb{Q}^2$ is dense in $X \times X$, $f(A)$ is nonempty. If A is a proper subset of L, then $L \setminus A$ is nonempty, and so V_A is nonempty. Thus $f(A)$ is not all of $\mathbb{Q}^2$. Now suppose A, B are subsets of L and $A \neq B$. WLOG let $x \in A \setminus B$. Thus $x \in U_A$ and $x \in V_B$, so $U_A \cap V_B$ is a nonempty open set, thus it contains an element of $\mathbb{Q}^2$, say q. Then $q \in f(A) \setminus f(B)$, so $f(A) \neq f(B)$.

However, f cannot be injective, since the cardinality of $\mathcal{P}(L)$ is strictly bigger than the cardinality of $\mathcal{P}(\mathbb{Q}^2)$.

See [Mun00, Example 3, p. 198] for a full exposition. //
Example 1.13 (Subspaces of normal need not be normal)**.** The following example is called Tychonoff's plank (see [SS78, Example 87, p. 106]). Let us take $X = \alpha\omega_1 \times \alpha\omega$. Recall that αX of a locally compact Hausdorff space X is the one-point compactification of it. Clearly X is compact and Hausdorff. Now, let us set Y to be $X \setminus (\infty_{\alpha\omega_1}, \infty_{\alpha\omega})$, so we remove the point added by the one point compactification. Let $A = \omega_1 \times \{\infty_{\alpha\omega}\}$, let $B = \{\infty_{\alpha\omega_1}\} \times \omega$. //

The following proposition gives sufficient conditions for normality. Similar theorems can be seen in [Mun00, Chp. 32, pp. 198–203].

[1] Previously proven

Proposition 1.14 (Sufficient conditions for normality). Let X be a space. Then X is normal if at least one of the conditions are satisified:

1. X is regular and Lindelof;
2. X is compact and Hausdorff;
3. X is metrizable;
4. X is a linearly ordered space (with the order topology).

Proof. (1) Suppose X is regular. Let A, B be closed and disjoint in X. For each $a \in A$, there are disjoint neighborhoods U_a of a and V_a of B. Since X is regular there is a neighborhood W_a of a such that $\overline{W_a} \subseteq U_a$. The collection of these W_a's is an open cover of A, and since A is closed, it too is Lindelof; thus it has a countable subcover. Let us denote this subcover by $\{ W_i \}$. Observe that each W_i has closure that is disjoint from B. Let us now repeat this construction for B, so that we have a countable collection of sets $\{ V_i \}$ such that V_i has closure disjoint from A.

We now set $W_n' = W_n \setminus \bigcup_{i=1}^n \overline{V_i}$ and $V_n' = V_n \setminus \bigcup_{i=1}^n \overline{W_i}$. Then each W_n' and V_n' are open. The collections $\{ W_n' \}$ and $\{ V_n' \}$ still cover A and B, since if $a \in A$, then $a \in W_n$ for some n, and a is not in $\overline{V_i}$ for all i as $\overline{V_i}$ is disjoint from A, thus $a \in W_n'$. A similar idea holds for B. Set $W = \bigcup W_n'$ and $V = \bigcup V_n'$. We are basically done; all that's left is to show W, V are disjoint. Suppose not, then there is some point $x \in W_n'$ and $x \in V_m'$ for some n, m. Without loss of generality, say $n \leq m$. Notice that by construction V_m' could not possibly contain any points of W_n', yet it does.

(2) See Proposition 5.4 in Week 8 Notes.

(3) To do this, we shall use the converse of Lemma 1.15. Let A, B be disjoint closed sets in X. Define

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)},$$

where $d(x, A) = \inf \{ d(x, a) : a \in A \}$. It is easy to see that f is continuous, and that f is 0 on A and 1 on B. ☐

Lemma 1.15 (Urysohn's). Let X be a normal topological space and A, B be disjoint closed sets in X. Then, there exists a continuous function $f : X \to [0, 1]$ such that f is 0 on A and f is 1 on B.

Proof. We shall construct for each rational number r, an open set U_r. These sets will have the following properties:

1. $U_r = \varnothing$ if $r < 0$, $U_r = X$ if $r > 1$;
2. $U_0 \supseteq A$ and $U_1 = X \setminus B$;
3. If $p < q$, then $\overline{U}_p \subseteq U_q$.

Let us first define $U_1 = X \setminus B$, and U_r for the cases in (1). To find U_0, we apply Lemma 1.10 to find U_0 such that $A \subseteq U_0$ and $\overline{U}_0 \subseteq U_1$, since U_1 contains A. Thus we have satisified condition (2).

At this point, it is unclear how we can get condition (3) out. What we would like to do; say r is a rational between p and q; is to shove the set U_r between U_p and U_q. We can do that due to normality: choose U_r such that $\overline{U_p} \subseteq U_r$ and $U_r \subseteq U_q$. If the rationals were well ordered in a way like the naturals, that would be easy. Unfortunately, reality is often disappointing. However, we observe crucially that if we had a finite collection of rationals and sets with property (3), we can do it quite easily. So instead we shall perform induction on a sequence containing all the rational numbers. Let $(r_i)_{i \in \mathbb{N}}$ be a sequence that enumerates all the rational numbers in $(0, 1)$ *exactly once*. By normality, there exists

U_{r_1} such that $\overline{U_{r_1}} \subseteq U_1$ and $\overline{U_0} \subseteq U_{r_1}$. Now, let $n \in \mathbb{N}$ and suppose we have chosen sets U_{r_i} where $i < n$ with the property that whenever $r_i < r_j$, then $\overline{U_{r_i}} \subseteq U_{r_j}$. Let p be the smallest rational number in the set $\{0, r_1, \ldots, r_{n-1}, 1\}$ that is bigger than r_n and q be the largest rational number from that set that is smaller than r_n. We quickly remark that this means $q < r_n < p$. By the inductive hypothesis, $\overline{U_q} \subseteq U_p$. Normality implies that there is a open set U_{r_n} such that $\overline{U_q} \subseteq U_{r_n}$ and $\overline{U_{r_n}} \subseteq U_p$.

Now, define

$$f(x) = \inf \{ q \in \mathbb{Q} : x \in U_q \}.$$

This function is well defined, and attains values between 0 and 1 due to property 1. Property 2 tells us that it is 0 on A and 1 on B. To show f is continuous, we shall show that preimages of subbasic elements are open, i.e. $f^{-1}[(-\infty, a)]$ and $f^{-1}[(a, \infty)]$ are open. To begin, we make the following observations:

$$f(x) < a \iff x \in U_p \text{ for some } p \in \mathbb{Q}, p < a. \tag{1}$$

$$f(x) \leq a \iff x \in \overline{U_p} \text{ for all } p \in \mathbb{Q}, p > a. \tag{2}$$

The first one (Equation (1)) follows immediately by definition of inf. For Equation (2), if $f(x) \leq a$, and $r > a$ is a rational, by definition of f, there is some rational $s < r$ such that $x \in U_s \subseteq U_r \subseteq \overline{U_r}$. Now for the converse. Suppose $s > a$ is a rational, we show that $f(x) \leq s$. Let us choose a rational r such that $a < r < s$. Then $x \in \overline{U_r}$ by hypothesis, and $\overline{U_r} \subseteq U_s$, so this tells us that $f(x) \leq s$. Since this holds for every rational bigger than a, we must have $f(x) \leq a$. Now, we thus have $f^{-1}[(-\infty, a)] = \bigcup_{r \in \mathbb{Q}, r < a} U_r$ and $f^{-1}[(a, \infty)] = X \setminus \bigcap_{r \in \mathbb{Q}, r > a} \overline{U_r}$. Both of those are open, so f is continuous. $\square$

The converse of Urysohn's lemma is true: if disjoint closed sets can be separated with continuous functions, then the space is normal.

2 Urysohn Metrization Theorem

Lemma 2.1. Let X be a T_1 space and let $\{ f_\alpha : \alpha \in \Lambda \}$ be a family of continuous functions from X into $[0, 1]$ such that for every $x \in X$ and every neighborhood U of x, there is some $\alpha \in \Lambda$ such that $f_\alpha(x) = 1$ and $f_\alpha|_{X \setminus U} = 0$. Then the map $F : X \to [0, 1]^\Lambda$ (product topology) given by $F(x) := \langle f_\alpha(x) : \alpha \in \Lambda \rangle$ is a topological embedding.

Proof. We first check injectivity. If $x \neq y$, there are neighborhoods U of x and V of y such that y is not in U and x is not in V. Thus there is some α such that $f_\alpha(x) = 1$ and f_α is 0 on $X \setminus U$, so in particular $f_\alpha(y) = 0$. So $F(x) \neq F(y)$.

Now, F is clearly continuous since every component of F is continuous[2]. Let us suppose U is open in X. Let $x \in U$, so that U is an open neighborhood of x. Thus there is some α such that $f_\alpha(x) = 1$ and f_α is 0 on $X \setminus U$. Let $V = \pi_\alpha^{-1}[(0, 1]]$. It remains to show that for every $F(z) \in F[U]$, there is a neighborhood of $F(z)$ living in $F[U]$. We claim that $F(z) \in F[X] \cap V \subseteq F[U]$, so $F[X] \cap V$ is our desired neighborhood. If $F[y] \in F[X] \cap V$ then $f_\alpha(y) > 0$, so $y \in U$, thus $F(y) \in F[U]$. $\square$

Corollary 2.2. *The following are equivalent:*

(1) X is completely regular;

(2) X is homeomorphic to a subspace of $[0, 1]^\Lambda$ for some Λ.

See [Mun00, Thm 34.3, p. 218].

Proof. Suppose X is completely regular. Then for each point x and neighborhood U of x, there is a continuous function $f_{x,U}$ which is 1 on x and 0 on $X \setminus U$, which is a closed set. The collection of all these functions satisfy the previous lemma. For the converse direction apply Proposition 1.11 (the closed unit interval is completely regular). $\square$

[2] This is the universal property of the product topology

Theorem 2.3 (Urysohn Metrization Theorem). Let X be a regular second-countable space. Then X is metrizable.

Proof. Since X is regular and second-countable, it is normal by Proposition 1.14 (second countability implies Lindelof). The idea here is to make the Λ in the previous lemma countable, then we would see that X is homeomorphic to a subspace of a metrizable space. What we need is a countable family of functions that separate points from closed sets. Let $\mathcal{B} = \{ U_n : n \in \mathbb{N} \}$ be a countable basis of X. For every n, m such that $\overline{U_n} \subseteq U_m$, using Lemma 1.15, choose a function $f_{n,m} : X \to [0,1]$ such that $f_{n,m}$ is 1 on $\overline{U_n}$ and it is 0 on $X \setminus U_m$. Let us check that these functions satisfy the property in the lemma. Suppose we have x and a neighborhood U of x. There is some basic U_m such that $x \in U_m \subseteq U$. By regularity, there is some U_n such that $x \in \overline{U_n} \subseteq U_m$. Then $f_{n,m}$ is 1 on $\overline{U_n}$, and it is 0 on $X \setminus U_m$, so in particular it is 0 outside of U. □

3 Stone-Čech Compactifications

We have previously constructed the Stone-Čech compactification of the naturals. Now we describe a different way to construct it that does not involve ultrafilters.

Theorem 3.1 (Existence of Stone-Čech Compactification). Let X be a space. Then, the following are equivalent:

1. X is completely regular;
2. X is homeomorphic to a subspace of $[0,1]^\Lambda$ for some Λ;
3. There is a compactification Y of X such that for every compact Hausdorff space K, every continuous function $f : X \to K$ has a unique continuous extension $g : Y \to K$.

Proof. Suppose X is completely regular. Let $(f_\alpha)_{\alpha\in\Lambda}$ be *all* the continuous functions from X into $[0,1]$. By the imbedding lemma, the function $F : X \to [0,1]^\Lambda$ that sends a point x to $\langle f_\alpha(x) : \alpha \in \Lambda \rangle$ is an imbedding. Let $Y = \overline{F[X]}$. Then Y is compact Hausdorff since it is a closed subspace of a compact space. Moreover, it is a compactification since X is dense in Y[3]. Let $f : X \to [0,1]$ be continuous, so there is some β such that $f = f_\beta$. Notice that $\pi_\beta : Y \to [0,1]$ is continuous, and $\pi_\beta(F(x)) = f_\beta(x) = f(x)$. If there is another function $g : Y \to [0,1]$ that extends f, notice that g agrees with π_β on $f[X]$, and so in particular it agrees with π_β on $\overline{f[X]} = Y$. So $g = \pi_\beta$. □

The situation of 3 is easily illustrated in the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook, "F"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\exists! g"] \\ & K \end{tikzcd}$$

We shall denote the Stone-Čech compactification of a space X by βX. Note that any other construction of this compactification is homeomorphic by universal property. Note additionally that we have only proven that any continuous function from X to $[0,1]$ factors uniquely through βX. However, the general case easily follows from the fact that K is compact Hausdorff if and only if it is homeomorphic to some closed space of $[0,1]^\Lambda$ for some Λ.

Proposition 3.2 (Properties of Stone-Čech compactification). Let X be completely regular. Then:

1. βX is unique up to homeomorphism;
2. βX is projectively maximal;
3. If X is locally compact Hausdorff; then αX (one-point compactification) is projectively minimal.

Proof. For (1), suppose Y_1, Y_2 are both Stone-Čech compactifications of X, so they both satisfy the extension property. Consider the following commutative diagrams.

[3] $F[X]$ is a homeomorphic copy of X in Y

$$\begin{tikzcd} X \arrow[r, hook, "i_2"] \arrow[dr, "i_1"'] & Y_2 \arrow[d, dashed, "\exists! \tilde{i_1}"] \\ & Y_1 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook, "i_1"] \arrow[dr, "i_2"'] & Y_1 \arrow[d, dashed, "\exists! \tilde{i_2}"] \\ & Y_2 \end{tikzcd}$$

Then, we have the following situation.

$$\begin{tikzcd} X \arrow[r, tail, "i_1"] \arrow[dr, tail, "i_1"'] & Y_1 \arrow[d, "\tilde{i_1} \circ \tilde{i_2}"'] \arrow[d, bend left, "\mathrm{Id}"] \\ & Y_1 \end{tikzcd}$$

By uniqueness, it must be that $\tilde{i_1} \circ \tilde{i_2} = \mathrm{Id}$. The other situation follows similarly[4].

For (2), we would like to show that if Y is any other compactification of X, then there is a continuous surjection $f : \beta X \to Y$ such that

- X is fixed pointwise (i.e. the restriction onto X is the identity);
- The topology of Y is the quotient topology induced by f.

So we may interpret this as saying that any other compactification of Y is a quotient of βX. Suppose Y is a compactification of X with the inclusion i_Y given. As usual, we apply the extension property. This immediately yields a unique continuous $f : \beta X \to Y$ such that $f \circ i_{\beta X} = i_Y$.

$$\begin{tikzcd} \beta X \arrow[dr, dashed, "\exists! f"] & \\ X \arrow[u, tail, "i_{\beta X}"] \arrow[r, tail, "i_Y"'] & Y \end{tikzcd}$$

Observe that $\overline{f[\beta X]} = Y$, since βX contains X and X is dense in Y. Additionally, $f[\beta X]$ is closed, since βX is compact and Y is Hausdorff. This shows that f is surjective. Let us now check that the topology on Y is indeed the quotient topology. Recall that the quotient topology on Y has the property that $U \subseteq Y$ is open if and only if $f^{-1}[U]$ is open. One direction is clear due to continuity. For the other direction, we shall consider closed sets. Let $C \subseteq Y$ and suppose that $f^{-1}[C]$ is closed in βX. It is thus compact. Applying f to this preimage, we note that $f\big[f^{-1}[C]\big] = C$. Since $f^{-1}[C]$ is compact the image under f of this set is compact, and thus closed in βX.

(3) Let Y be a compactification of X, let $f : Y \to \alpha X$ be a surjection defined by the following:

$$f(y) = \begin{cases} x & \text{if } x \in X, \\ \infty & \text{if } y \in Y \setminus X. \end{cases}$$

We need to show that the topology on αX is the quotient topology induced by f. It will suffice to show f is continuous and apply the same argument as above, when we showed that Y has the quotient topology induced by f. There are 2 kinds of open sets: $U \subseteq X$ (does not contain the point ∞) and $U \subseteq \alpha X$ (contains ∞). Suppose $U \subseteq X$ is open, then the preimage under f of U is just $U \subseteq Y$. Now by Lemma 3.3, U is open in Y. For the second kind, if $U \subseteq \alpha X$, we know that $X \setminus U$ is compact. Now, $f^{-1}[U]$ has compact complement in Y, thus it is open. $\square$

Lemma 3.3. If X is locally compact Hausdorff, then X is an open subset of any of its compactifications.

Proof. Let Y be a compactification of X. Let $x \in X$. We shall find a neighborhood of x contained in X. Since X is locally compact Hausdorff, there is a neighborhood U of x, open in X such that $\overline{U} \subseteq X$ and $\overline{U}$ is compact[5]. Since X is a subspace of Y, there is some V open in Y such that $U = V \cap X$. This V will be the desired neighborhood. We claim that $V \subseteq \overline{V \cap X}$. To see this, if $b \in V$ and W is any neighborhood of b, there is some $a \in X$ such that $a \in V \cap W$ as X is dense. So every neighborhood around b contains a point of X and V. Thus $b \in \overline{V \cap X}$. $\square$

[4] We remark that this can be used in a more general setting to prove that universal properties characterize objects uniquely up to isomorphism in any category.

[5] We won't be needing the compactness, only the fact that the closure stays in X.

Interestingly enough, $\beta\omega$ is $\alpha\omega$. More surprisingly, $\beta\omega_1$ is $\alpha\omega_1$.