Week 10

Robert

July 2024

List of Definitions

List of Propositions

1 Local compactness and one-point compactifications

Definition 1.1 (Local Compactness). A space X is **locally compact at** $x \in X$ if there is a neighborhood U of x and a compact set K containing U.

A space is locally compact if it is locally compact at every point.

Clearly every compact space is locally compact. We now exhibit some examples of spaces which are not compact, but locally compact. We also exhibit examples of spaces that are not locally compact.

Example 1.2 (The real numbers). The real numbers are locally compact. Given $x \in \mathbb{R}$, take an interval $(x-\varepsilon, x+\varepsilon)$ around it and the closed interval $[x-\varepsilon, x+\varepsilon]$ contains this neighborhood. //

Example 1.3 (The rational numbers). The rational numbers are *not* locally compact everywhere. If $\mathbb{Q}$ was locally compact, pick $q \in \mathbb{Q}$, then there is a neighborhood U of q and compact subspace C such that $q \in U \subseteq C$. Now, U contains some interval $(q-\varepsilon, q+\varepsilon)$. Pick an irrational number in this interval. Then choose a sequence of rational numbers converging to this irrational number[1]. The limit should be in C as C is closed, but the limit is irrational.

In fact, every closed interval $[a,b] \cap \mathbb{Q}$ is seen to be not compact, since $\{ (a-1, i) : i \text{ irrational }, a < i < b \}$ is an open cover of $[a,b] \cap \mathbb{Q}$[2] with no finite subcover. //

When the space X is Hausdorff, we can say a bit more.

Proposition 1.4 (Equivalent conditions for local compactness in Hausdorff spaces). *Let X be Hausdorff. Then, the following are equivalent:*

1. *X is locally compact at x;*
2. *There is a neighborhood V of x such that $\overline{V}$ is compact;*
3. *There is a local basis around x of open sets with compact closure.*

Proof. It is immediate that 1 implies 2, since $\overline{U} \subseteq K$ as K is closed and contains U. Moreover, $\overline{U}$ is closed so it is compact. For 2 implies 3, set $\mathcal{B}_x = \{ U \cap V : U \text{ is a neighborhood of } x \}$ where V is given by (2). Then, $\overline{U \cap V} \subseteq \overline{V}$, so the sets $U \cap V$ have compact closure. 3 immediately implies 1. ☐

Example 1.5 (Discrete topology). Any space with the discrete topology is easily seen to be locally compact. //

Theorem 1.6 (Existence of one-point compactifications). Let X be a space. Then, the following are equivalent:

1. X is Hausdorff and locally compact;
2. There is a compact Hausdorff space Y and an embedding $i : X \to Y$ such that $Y \setminus i[X]$ is a single point;
3. X is homeomorphic to an open subspace of a compact Hausdorff space.

Moreover, if there is another space Y' and embedding j that satisfies (2), then there is a unique homeomorphism $f : Y \to Y'$ such that $f \circ i = j$.[a]

[a]Warning: This is *not* a universal property.

See [Mun00, Thm 29.1, p. 181].

Proof. Suppose X is locally compact Hausdorff with topology τ. Let[3] $\infty \notin X$. Define $Y = X \cup \{ \infty \}$ and we shall define a topology τ_∞ on Y with the following properties:

- Everything that is open in X should be open in Y, i.e. $\tau \subseteq \tau_\infty$;
- We also add neighborhoods of ∞, they should be subsets U of Y such that $X \setminus U$ is compact in X. Thus neighborhoods of ∞ have compact complement in X.

Let us check that τ_∞ is a topology. Notice $\varnothing \in \tau_\infty$ since it is open in X. We also have $Y \in \tau_\infty$.

[1]If i is such an irrational, pick x_n to be rational such that $i < x_n < i + 1/n$.
[2]We are covering it in $\mathbb{R}$, since compactness is not relative.
[3]You can choose $\infty = X$ since $X \notin X$.

If $U, V \subseteq X$ then $U \cap V$ are open in X and so open in Y. If U, V are open neighborhoods of ∞ then we have $X \setminus U = K_U, X \setminus V = K_V$ being compact. Then we have $X \setminus (U \cap V) = K_U \cup K_V$ which is compact. If $U \subsetneq X$ and V an open neighborhood of ∞, then $X \setminus V$ is compact. Then $(X \cap V) \cap U$ is open. Unions are left as an exercise.

For compactness, let $\mathcal{U}$ be an open cover of Y. Let $U \in \mathcal{U}$ be such that $\infty \in U$. Then $X \setminus U$ is compact and so it has a finite subcover from $\mathcal{U}$.

To see Y is Hausdorff, pick $x, y \in Y$. If both of x, y lie in X it is easy. If $y = \infty$, use the local compactness of X to choose a neighborhood U of x with a compact subspace C containing U. Then $Y \setminus C$ is a neighborhood of $y = \infty$ and it is disjoint from U.

2 implies 3 follows because the singleton ∞ is closed in Y.

For 3 implies 1, it is immediate that X is Hausdorff. Let $x \in X$. Then there is some open neighborhood U of x such that $\overline{U} \subseteq X$[4]. Then $\overline{U}$ is compact and contains U.

The uniqueness condition is easy. $\square$

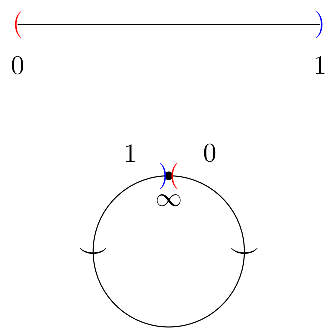

Figure 1: One point compactification of the reals. The reals are homeomorphic to $(0, 1)$.

Remark 1.7. Notice that we can construct a one point compactification of a space that is not necessarily locally compact or even Hausdorff.

Definition 1.8 (Compactification of Hausdorff space)**.** A **compactification** of a Hausdorff space X is a compact Hausdorff space Y such that X is dense in Y. If $Y \setminus X$ is a single point, then Y is called the **one-point compactification** of X, and it is denoted αX.

Although X may not actually be a subset of Y, we mean that there is an embedding $i : X \to Y$. Also, $Y \setminus X$ really means $Y \setminus i[X]$.

Remark 1.9. Constructing a one compactification of X when X is already compact is uninteresting since it simply adjoins an isolated point to X.

Remark 1.10. If X is locally compact Hausdorff and we have a construction Y that we claim to be the one point compactification of X, it suffices to show that Y is compact Hausdorff, $Y \setminus X$ is a singleton, and X embeds into Y as a dense subset. This is due to the uniqueness condition of Theorem 1.6.

Example 1.11 (One-point compactification of the naturals)**.** It is claimed that $\alpha\mathbb{N}$ is homeomorphic to $\{ 1/n : n \in \mathbb{N} \} \cup \{ 0 \}$. //

Example 1.12 (One-point compactification of the reals)**.** From Figure 1, you can visualize $\alpha\mathbb{R}$ as the circle. They are homeomorphic. //

Example 1.13 (One-point compactification of ω_1)**.** This has been discussed in Problem set 3, Question 8, slightly. //

2 Other compactness notions

Recall that a limit point of a set A is a point p such that every neighborhood of p intersects A at some point other than p.

[4]See [Lee11, Lem 4.65] or Question 4a in Problem Set 7

Definition 2.1 (Limit point compactness)**.** A space is **limit point comapct** if every infinite subset has a limit point.

Definition 2.2 (Sequential compactness)**.** A space is **sequentially compact** if every sequence has a convergent subsequence.

Definition 2.3 (Countable compactness)**.** A space is **countably compact** if every *countable* open cover of it has a finite subcover.

Proposition 2.4 (Countable compactness and limit point compactness)**.** *If X is countably compact then it is limit point compact. If X is a T_1 space then the converse is true.*

Proof. Suppose $A \subseteq X$ is an infinite subset with no limit points. We may assume A is countable, since if A is uncountable and has no limit points then no countable subset of A would have limit points. Then A is closed. For each $a \in A$, there is some open neighborhood $U_a \subseteq X$ such that U_a intersects A only at a. Let us cover the space X with $X \setminus A$, and the sets U_a. This is a countable cover, but it cannot have a finite subcover. If we remove a single U_a, then we will not cover all of A.

Now suppose X is T_1. Suppose that $(U_n)_{n\in\mathbb{N}}$ is a countable cover of $\mathbb{N}$ with no finite subcover. For each $n \in \mathbb{N}$, let us choose $x_n \in X \setminus \bigcup_{j\leq n} U_j$. Now we have an infinite set $A = \{ x_n \}$. If this was not infinite then there is some point we had to choose infinitely often, call it x_k. There is some open set U_m that includes x_k. Now let $j > m$ be such that we chose x_k again. This is impossible. For $x \in X$, let U_n be such that $x \in U_n$. Then, $U_n \cap A$ has to be finite. As such, the sequence (x_n) does not have a convergent subsequence. Now, $U_n \setminus (A \setminus \{ x \})$ is an open neighborhood of x (we are only removing finitely many points, and points are closed) and it contains no points of A. So A would have no limit point. □

Proposition 2.5 (Compactness implies limit point compactness)**.** *If X is compact then it is limit point compact.*

Proof. See Munkres. □

Example 2.6 (Limit point compact but not countably compact)**.** The following construction is called the *divisor topology*. Let $X = \{ n \in \mathbb{N} : n \geq 2 \}$ and declare a basis on X of open sets $U_n = \{ x \in \mathbb{N} : x \text{ divides } n \}$ for $n \geq 2$. This is a basis because if U_n intersects U_m at some x, then x divides both n and m so it divides their gcd. Thus $U_{\gcd(n,m)}$ will be contained in $U_n \cap U_m$. This is not countably compact, but it is limit point compact. The basis we declared covers X, but each U_n is finite and as such a finite number of them cannot cover X. However, it is limit point compact. Let $A \subseteq X$ be infinite, so we have 2 different elements in A, say $m, n \in A$. //

Example 2.7 (Compact but not sequentially compact)**.** The stone-cech compactification of the naturals, $\beta\mathbb{N}$, is compact but the only sequences that converge are eventually constant. Thus, it is not sequentially compact, since we can easily pick a sequence which has no subsequences that are eventually constant. //

Example 2.8 (Limit point compact but not compact)**.** The first uncountable ordinal, ω_1, is not compact. We can cover it with initial segments by $\mathcal{U} = \{ [0, \beta) : \beta \in \omega_1 \}$. Each initial segment is countable and we cannot extract any finite subcover, since any finite union of countable sets is countable. (This also shows it is not Lindeloff). It is however limit point compact. Let $A \subseteq \omega_1$ be countable, so that A is bounded[5]. Then $A \subseteq [0, \beta]$ where β is a bound for A, and $[0, \beta]$ is compact so A has a limit point in that closed interval which is compact. Then this limit point is also a limit point of A in ω_1. //

The previous example also goes to show that ω_1 is sequentially compact, but not compact.

Proposition 2.9 (Equivalence of compactness notions for metric spaces)**.** *For metric spaces, compactness, limit point compactness and sequential compactness are equivalent.*

3 Countability axioms

Definition 3.1 (First and second countability)**.** A topological space X is **first-countable** if every point has a countable local basis. It is **second-countable** if there is a countable basis of X.

[5] See problem set 2

Recall that a space is sequential if given $A \subseteq X$, and $x \in \overline{A}$, there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ such that $x_n \to x$.

Proposition 3.2 (First countable implies sequential)**.** *Let X be first-countable. Then, X is sequential.*

Proof. Suppose $x \in \overline{A}$. If $\{ U_n \}$ is a local basis at x, then pick $x_n \in A \cap \bigcap_1^n U_i$. □

Proposition 3.3 (Metric spaces are first countable)**.** *If X is metrizable, then X is first countable*

Proof. Choose balls of radius $1/n$ at each point x. □

Example 3.4 (The real numbers)**.** The real numbers are easily seen to be second countable by taking the basis $\mathcal{B} = \{ (p,q) : p < q, p, q \in \mathbb{Q} \}$. //

Proposition 3.5 (Restrictions on the cardinality of discrete subspaces)**.** *If X is second countable and $A \subseteq X$ is discrete, then A is countable.*

Proof. If A is discrete, we can put a basis element U_a around each point $a \in A$ that intersects A only at $\{ a \}$, so we have a collection of distinct basis elements of at least the cardinality[6] of A. □

Example 3.6 (Uniform topology)**.** The space $\mathbb{R}^\omega$ with the uniform topology is first countable but not second countable. It contains the subspace $\{ 0,1 \}^\omega$, the set of binary seqeunces, which is a discrete subspace. By Proposition 3.7, the uniform topology on $\mathbb{R}^\omega$ cannot be second countable. //

Proposition 3.7 (Implications of second countability)**.** *Let X be a second countable space. Then, the following are true:*

1. *X is first countable;*
2. *X satisifies the **Lindelof** property: Every open cover of X has a countable subcover;*
3. *X is **separable**[7]: There is a countable dense subset of X.*
4. *X has the **countable chain condition (ccc)**: Any collection of pairwise disjoint open subsets of X is countable.*

Proof. 1 is immediate by choosing a local basis of x to be the elements of the countable basis of X that contain x.

2 is an exercise

For 3, let $\mathcal{B}$ be a countable basis of X and pick[8] a $x \in U \in \mathcal{B}$ (for each U that is nonempty in $\mathcal{B}$). Then let D be the collection of all these points we picked. It is clear that D intersects every open set so it is dense.

4 follows by observing that you can extract a pairwise disjoint collection of basis elements from a pairwise disjoint collection of open subsets. Alternatively, you can deduce 4 from 3 by picking an element of the dense set from each open subset in the collection. □

Note none of these implications can be reversed, except for metric spaces.

Remark 3.8. Even with all 3 conditions being satisified, X may still fail to be second countable. See [Mun00, Example 3, p. 192].

Example 3.9. Endow $\mathbb{R}$ with the cocountable topology. Then this topolgy is Lindelof and has CCC, but it is not second countable. //

Example 3.10. Let $X = \mathbb{R}_l^2$ be the Sorgenfrey plane. Then it has a countable dense subset, namely $\mathbb{Q}^2$. A basic open set looks like $[a,b) \times [c,d)$ so you can easily find a point with both rational coordinates.[9] It is not Lindelof because we can find an uncountable discrete subspace. We will use the strategy of Proposition 2.4. Namely, let L be the line generated by $y = -x$. For each point $p \in L$ let U_p be a basic open set that intersects L only at p. Then cover the Sorgenfrey plane with the complement of L and the U_p's. No countable subcover can exist, since if we remove *any* of the U_p's we will not cover a point in the line. //

Note that $\mathbb{R}_l$ is actually Lindelof.

Example 3.11. Let $\alpha\omega_1$ be the one point compactification of ω_1. Then it is Lindelof since it is compact. It does not have CCC, since for every $\gamma \in \alpha\omega_1$ that has an immediate predecessor, $\{ \gamma \}$ is open. There are uncountably many γ with an immediate predecessor. It is also not separable. //

[6] In particular, the map $a \mapsto U_a$ from A into the basis of X is injective.

[7] This name is quite bad, since it really has nothing to do with separations and connectedness.

[8] The axiom of countable choice will suffice

[9] Recall that an equivalent condition for a set being dense is that every nonempty open set intersects it.

Proposition 3.12 (Equivalence of countability axioms for metric spaces)**.** *For metrizable spaces, second countability, possessing the Lindeloff property and separability are equivalent.*