Yoneda lemma notes

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Abstract

Personal notes on the proof of the Yoneda lemma. This is the contravariant version.

1 Notation

If $\mathbf{C}$ is a locally small category then we denote the hom set $\mathrm{Hom}_{\mathbf{C}}(x, y)$ by $\mathbf{C}(x, y)$.

The notation $\mathbf{C}(-, x)$ is for the contravariant representable functor which takes an object y to it's hom set

$$y \mapsto \mathbf{C}(y, x)$$

and a morphism $h : y \to z$ to the morphism of hom sets

$$\mathbf{C}(h, x) : \mathbf{C}(z, x) \to \mathbf{C}(y, x)$$

Where if $f \in \mathbf{C}(z, x)$, $\mathbf{C}(h, x)(f) = f \circ h$ (we call this precomposition by h)

Let $h : x \to y$. The notation $\mathbf{C}(-, h)$ is for a natural transformation of contravariant representable functors. Namely, $\mathbf{C}(-, h) : \mathbf{C}(-, x) \to \mathbf{C}(-, y)$.

Given 2 set-valued functors $F, G : \mathbf{C} \to \mathbf{Sets}$, we denote the set of natural transformations between them by $\mathrm{Nat}(F, G)$

2 The lemma

Theorem 2.1 (Yoneda Lemma)**.** Let $\mathbf{C}$ be a locally small category. Then, for any object $x \in \mathbf{C}$ and contravariant set-valued functor $F : \mathbf{C}^{\text{op}} \to \mathbf{Sets}$, there an isomorphism $\text{Nat}(\mathbf{C}(-, x), F) \cong Fx$. Moreover, this isomorphism is natural in F, meaning the diagram below commutes:

$$\begin{array}{ccc} \text{Nat}(\mathbf{C}(y, x), F) & \xrightarrow{\cong} & Fy \\ {\scriptstyle \text{Nat}(\mathbf{C}(y,x),\vartheta)}\downarrow & & \downarrow{\scriptstyle \vartheta_y} \\ \text{Nat}(\mathbf{C}(y, x), G) & \xrightarrow[\cong]{} & Gy \end{array}$$

and it is natural in x, meaning that

$$\begin{array}{ccc} \text{Nat}(\mathbf{C}(-, x), F) & \xrightarrow{\cong} & Fx \\ {\scriptstyle \text{Nat}(\mathbf{C}(-,h),F)}\uparrow & & \uparrow{\scriptstyle F(h)} \\ \text{Nat}(\mathbf{C}(-, y), F) & \xrightarrow{\cong} & Fy \end{array}$$

commutes given $h : x \to y$, a morphism in $\mathbf{C}$

It is highly recommended to pull out a pen and paper and follow along as there are MANY different mathematical objects here. We section the proof into 4 parts, namely,

1. Defining the isomorphism and checking it is well defined
2. Checking it is bijective
3. Checking it is natural in F
4. Checking it is natural in c

With that in mind, let's begin.

Construction of the isomorphism. Define

$$\eta_{x,F} : \text{Nat}(\mathbf{C}(-, x), F) \to Fx$$

as follows: Given a natural transformation $\vartheta \in \text{Nat}(\mathbf{C}(-, x), F)$, we define

$$\eta_{x,F}(\vartheta) = \vartheta_x(1_x) \tag{1}$$

Here, $\vartheta_x : \mathbf{C}(x, x) \to Fx$ is the morphism, and $1_x \in \mathbf{C}(x, x)$. Now define

$$\varphi_{x,F} : Fx \to \text{Nat}(\mathbf{C}(-, x), F)$$

by taking any $a \in Fx$ to the natural transformation $\psi_a : \mathbf{C}(-, x) \to F$ where each component of ψ_a, $(\psi_a)_z$ for $z \in \mathbf{C}$ to be $(\psi_a)_z : \mathbf{C}(z, x) \to Fz$, taking $h \in \mathbf{C}(z, x)$ to $F(h)(a)$. Symbolically,

$$(\psi_a)_z\,(h) = F(h)(a)$$

Checking that $\varphi_{x,F}$ is well defined is left as an exercise for the reader (just check that the natural transformation produced is in fact a natural transformation) $\square$

Proof of the bijection. We would like to check that $\varphi_{x,F} \circ \eta_{x,F}$ is indeed the identity on $\mathrm{Nat}\left(\mathbf{C}(-,x), F\right)$. Likewise, we need to check that $\eta_{x,F} \circ \varphi_{x,F}$ is the identity on Fx. Let's do the first one. Let $\vartheta \in \mathrm{Nat}\left(\mathbf{C}(-,x), F\right)$. Now,

$$\varphi_{x,F} \circ \eta_{x,F}(\vartheta) = \varphi_{x,F}(\vartheta_x(1_x)) = \psi_{\vartheta_x(1_x)}$$

Keep in mind that $\psi_{\vartheta_x(1_x)}$ is a natural transformation $\mathbf{C}(-,x) \to F$, where each component $\left(\psi_{\vartheta_x(1_x)}\right)_z$ is a morphism of homsets $\mathbf{C}(z,x) \to Fz$, and if $h \in \mathbf{C}(z,x)$ then

$$\left(\psi_{\vartheta_x(1_x)}\right)_z(h) = F(h)(\vartheta_x(1_x)) \tag{2}$$

Now, since ϑ is natural, for our $h \in \mathbf{C}(z,x)$, the following commutes:

$$\begin{array}{ccc} \mathbf{C}(z,x) & \xrightarrow{\vartheta_z} & Fz \\ {\scriptstyle \mathbf{C}(h,x)}\uparrow & & \uparrow{\scriptstyle F(h)} \\ \mathbf{C}(x,x) & \xrightarrow[\vartheta_x]{} & Fx \end{array}$$

Now, let's choose the identity morphism $1_x \in \mathbf{C}(x,x)$. Since the diagram commutes, we know that $\left(\vartheta_z \circ \mathbf{C}(h,x)\right)(1_x) = \left(F(h) \circ \vartheta_x\right)(1_x)$. Referring back to Equation (2) we can see that the right side is exactly $\left(\psi_{\vartheta_x(1_x)}\right)_z(h)$. Now let's see what the left side is. Firstly, $\mathbf{C}(h,x)(1_x) = 1_x \circ h = h$. Now this means that $\left(\vartheta_z \circ \mathbf{C}(h,x)\right)(1_x) = \vartheta_z(h)$. Since h was arbitrary $\left(\psi_{\vartheta_x(1_x)}\right)_z = \vartheta_z$. Since z was also arbitrary this means $\psi_{\vartheta_x(1_x)} = \vartheta$.

Now let's do the next one. This one is easier. Recall that at this point we wish to check that $\eta_{x,F} \circ \varphi_{x,F}$ is the identity on Fx. Let $a \in Fx$ be arbitrary. By definition, $\varphi_{x,F}(a) = \psi_a$. Now by definition again $\eta_{x,F}(\psi_a) = (\psi_a)_x(1_x)$. Recall that $(\psi_a)_x$ takes $h : x \to x$ to $F(h)(a)$. Now, this means $(\psi_a)_x(1_x) = F(1_x)(a)$. Since F is a functor $F(1_x)$ is the identity on Fx, so $F(1_x)(a) = a$ as desired. $\square$

Proof of naturality in F. Let $\phi : F \to G$ be a natural transformation. We would like to prove that

$$\begin{array}{ccc} \mathrm{Nat}(\mathbf{C}(-,c), F) & \xrightarrow{\eta_{c,F}} & Fc \\ {\scriptstyle \mathrm{Nat}(\mathbf{C}(-,c),\phi)}\downarrow & & \downarrow{\scriptstyle \phi_c} \\ \mathrm{Nat}(\mathbf{C}(-,c), G) & \xrightarrow[\eta_{c,G}]{} & Gc \end{array}$$

commutes. Again, recall that the morphism $\mathrm{Nat}\left(\mathbf{C}(-,c), \phi\right)$ simply takes any $\vartheta \in \mathrm{Nat}\left(\mathbf{C}(-,c), F\right)$ and composes it with ϕ, that is $\vartheta \mapsto \phi \circ \vartheta$. Now let's check this.

Let $\vartheta \in \text{Nat}\left(\mathbf{C}(-,c), F\right)$ be arbitrary. Now,

$$\begin{aligned}
&\left(\phi_c \circ \eta_{c,F}\right)(\vartheta) \\
&= \phi_c(\eta_{c,F}(\vartheta)) \\
&= \phi_c(\vartheta_c(1_c)) && \text{By Equation (1)}
\end{aligned}$$

By how composition of natural transformations is defined, $\phi_c \circ \vartheta_c = (\phi \circ \vartheta)_c$. So this means that $\phi_c(\vartheta_c(1_c)) = (\phi \circ \vartheta)_c\,(1_c)$. The natural transformation $\phi \circ \vartheta$ has codomain G, since ϕ is a natural transformation from F to G. Now by definition of $\eta_{c,G}$ we know that $(\phi \circ \vartheta)_c\,(1_c) = \eta_{c,G}\,(\phi \circ \vartheta)$. Notice that $\phi \circ \vartheta$ is really just $\text{Nat}(\mathbf{C}(-,c), \phi)(\vartheta)$. So combining all this together, we have $\eta_{c,G}\left(\text{Nat}\left(\mathbf{C}(-,c), \phi\right)(\vartheta)\right)$. Since ϑ was arbitrary the diagram commutes. □

Proof of naturality in c. Let $h : x \to y$ be a morphism in $\mathbf{C}$. We would like to show that

$$\begin{array}{ccc}
\text{Nat}(\mathbf{C}(-,x), F) & \xrightarrow{\eta_{x,F}} & Fx \\
{\scriptstyle \text{Nat}(\mathbf{C}(-,h),F)}\uparrow & & \uparrow{\scriptstyle F(h)} \\
\text{Nat}(\mathbf{C}(-,y), F) & \xrightarrow{\eta_{y,F}} & Fy
\end{array}$$

commutes.

Let $\vartheta \in \text{Nat}\left(\mathbf{C}(-,y), F\right)$ be arbitrary. Following the red path, $\left(F(h) \circ \eta_{y,F}\right)(\vartheta) = F(h)\left(\vartheta_y(1_y)\right)$.

By naturality of ϑ,

$$\begin{array}{ccc}
\mathbf{C}(x,y) & \xrightarrow{\vartheta_x} & Fx \\
{\scriptstyle \mathbf{C}(h,y)}\uparrow & & \uparrow{\scriptstyle F(h)} \\
\mathbf{C}(y,y) & \xrightarrow{\vartheta_y} & Fy
\end{array}$$

So

$$\begin{aligned}
F(h)\left(\vartheta_y(1_y)\right) &= \left(\vartheta_x \circ \mathbf{C}(h,y)\right)(1_y) \\
&= \vartheta_x(1_y \circ h) && \text{By definition of } \mathbf{C}(h,y) \\
&= \vartheta_x(h)
\end{aligned}$$

Keep in mind that $\text{Nat}\left(\mathbf{C}(-,h), F\right)$ precomposes ϑ with $\mathbf{C}(-,h)$. That is, $\vartheta \mapsto \vartheta \circ \mathbf{C}(-,h)$. Also, $\mathbf{C}(-,h)_a$ is a morphism $\mathbf{C}(a,x) \to \mathbf{C}(a,y)$, which takes a morphism $f \in \mathbf{C}(a,x)$ and composes it with h, that is $f \mapsto h \circ f$.

Now following the blue path,

$$
\begin{aligned}
& \left(\eta_{x,F} \circ \operatorname{Nat}\left(\mathbf{C}(-,h), F\right)\right)(\vartheta) & \\
&= \eta_{x,F}(\vartheta \circ \mathbf{C}(-,h)) && \text{See above paragraph} \\
&= \left(\vartheta \circ \mathbf{C}(-,h)\right)_x (1_x) && \text{By definition of } \eta_{x,F}\text{, see Equation (1)} \\
&= \left(\vartheta_x \circ \mathbf{C}(-,h)_x\right)(1_x) && \text{Composition of natural transformations} \\
&= \vartheta_x \left(\mathbf{C}(-,h)_x(1_x)\right) && \\
&= \vartheta_x \left(h \circ 1_x\right) && \text{Definition of } \mathbf{C}(-,h)_x \\
&= \vartheta_x (h) &&
\end{aligned}
$$

This completes the proof. □