# Chapter 9 Summary

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## 1 Adjoints

**Definition 1.1.** Let $\mathbf{C}, \mathbf{D}$ be categories, and let $F : \mathbf{C} \to D$ and $G : \mathbf{D} \to \mathbf{C}$ be functors. Then $F$ is left adjoint to $G$ and $G$ is right adjoint to $F$, if we have the following situation:

$$\mathbf{D}(Fc, d) \cong \mathbf{C}(c, Gd)$$

(given objects $c \in \mathbf{C}$ and $d \in \mathbf{D}$). The isomorphism also has to be natural in $c, d$.

Intuitively it feels like adjoint functors have some kind of symmetry to them.

**Example 1.2** (Riehl, Category Theory in Context)**.** Let $U : \mathbf{Top} \to \mathbf{Sets}$ be the forgetful functor. Then $U$ has a left and right adjoint. For a left adjoint of $U$, we need some functor $L : \mathbf{Sets} \to \mathbf{Top}$ such that $\mathbf{Sets}(UX, S) \cong \mathbf{Top}(X, LS)$ naturally. So we need to make a topological space from $S$ such that every function $f : U(X) \to S$ corresponds to exactly one continuous function $\tilde{f} : X \to L(S)$. Since we know that any function out of a space with the discrete topology is continuous, we can let $L$ take a set $S$ to the topological space $(S, \mathcal{P}(S))$.

For the right adjoint, we can let $R$ take a set $S$ to $(S, \{\varnothing, S\})$, endowing it with the trivial topology. Since every function into a space with the trivial topology is continuous $U, R$ are indeed adjoint. //

Forgetful functors often admit left adjoints. For example the functor taking a set to the free monoid on it is left adjoint to the forgetful functor from monoids to sets. The functor that takes a set to its free abelian group is also left adjoint to the forgetful functor from abelian groups to sets.

Adjoints can also be equivalently characterized by the following:

**Definition 1.3.** Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be functors. Then $F$ is left adjoint to $U$ if there is a natural transformation $\eta : 1_{\mathbf{C}} \to G \circ F$ such that given any $c \in \mathbf{C}, d \in \mathbf{D}$, $f : c \to G(d) \in \operatorname{Mor} \mathbf{C}$, there is a unique $g : Fc \to d \in \operatorname{Mor} \mathbf{D}$ such that $f = G(g) \circ \eta_c$.

Here $\eta$ is called the unit, and if we have $\phi : \mathbf{D}(Fc, d) \cong \mathbf{C}(c, Gd)$ natural, then we can obtain $\eta_c = \phi(1_{Fc})$, and $\phi(g) = G(g) \circ \eta_c$.

By duality, we can also get a counit, a natural trasnformation $\varepsilon : F \circ G \to 1_{\mathbf{D}}$ such that given any $c \in \mathbf{C}, d \in \mathbf{D}$, $g : Fc \to d \in \operatorname{Mor} \mathbf{D}$, there is a unique $f : c \to Gd \in \operatorname{Mor} \mathbf{C}$ such that $g = \varepsilon_d \circ F(f)$.

**Example 1.4.** Let $\mathbf{C}$ be a category with binary products. Fix an object $A \in \mathbf{C}$. Then $(-) \times A$ is left adjoint to the exponential functor $(-)^A$. The counit here is what gives us the evaluation morphism. //

If a functor admits a left/right adjoint it is a natural question to wonder whether said left/right adjoint is unique. It turns out that they are.

**Proposition 1.5** (Uniqueness of adjoints)**.** *If $L : \mathbf{C} \to D$ has right adjoints $R, S : \mathbf{D} \to C$ then $R \cong S$. By symmetry this is also true for left adjoints.*

The proof of this proposition follows from applying the Yoneda principle to the fact that $\mathbf{C}(c, Rd) \cong \mathbf{C}(c, Sd)$ naturally.

There are also the adjoint functor theorems but I feel like I'm going to need quite a bit more time to understand it.