We define the Yoneda embedding (contravariant version) to be as follows:

$$\mathcal{Y} : \mathbf{C} \to \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Sets})$$
$$\mathcal{Y}(x) = \mathbf{C}(-, x)$$
$$\mathcal{Y}(f : x \to y) = \mathbf{C}(-, f) : \mathbf{C}(-, x) \to \mathbf{C}(-, y)$$

Sending an object to a representable functor, and a morphism to a natural transformation of representable functors.

Theorem 0.1 (Yoneda Lemma). Let $\mathbf{C}$ be a locally small category. Then, for any object $x \in \mathbf{C}$ and contravariant set-valued functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$, there an isomorphism $\mathrm{Nat}(\mathbf{C}(-, x), F) \cong Fx$. Moreover, this isomorphism is natural in F, meaning the diagram below commutes:

$$\begin{tikzcd}
\mathrm{Nat}(\mathbf{C}(y,x),F) \arrow[r, "\cong"] \arrow[d, "\mathrm{Nat}(\mathbf{C}(y,x),\vartheta)"'] & Fy \arrow[d, "\vartheta_y"] \\
\mathrm{Nat}(\mathbf{C}(y,x),G) \arrow[r, "\cong"'] & Gy
\end{tikzcd}$$

and it is natural in x, meaning that

$$\begin{tikzcd}
\mathrm{Nat}(\mathbf{C}(-,x),F) \arrow[r, "\cong"] & Fx \\
\mathrm{Nat}(\mathbf{C}(-,y),F) \arrow[r, "\cong"] \arrow[u, "\mathrm{Nat}(\mathbf{C}(-,h),F)"] & Fy \arrow[u, "F(h)"']
\end{tikzcd}$$

commutes given $h : x \to y$, a morphism in $\mathbf{C}$

There is also a covariant version of the Yoneda lemma, where instead of looking at the natural transformations between $\mathbf{C}(-, x)$ and F (contravariant) we look at the natural transformations between $\mathbf{C}(x, -)$ and F (covariant).

The Yoneda lemma actually tells us a few things. The main one being that given some set-valued functor F, we know for sure that natural transformations from representable functors to F are a set, and every single one can be classified by simply studying Fx.

Remark 0.2. The actual explicit isomorphism is important. The isomorphism $\mathrm{Nat}(\mathbf{C}(-, x), F) \to Fx$ is defined by sending $\vartheta \in \mathrm{Nat}(\mathbf{C}(-, x), F)$ to $\vartheta_x(1_x)$ given by evaluating the xth component of ϑ on the identity morphism. The inverse is given by taking $a \in Fx$ to the natural transformation ψ_a defined componentwise for each $z \in \mathbf{C}$ as $(\psi_a)_z(h) = F(h)(a)$

Remark 0.3. In Riehl's book we have that a functor F is represented by a object x if there is a natural isomorphism $\mathbf{C}(x, -) \cong F$ (or the contravariant version). I wonder if the Yoneda lemma can actually be used to extract the representing object.

We may rephrase the Yoneda principle as the following corollary:

Corollary 0.4. *Suppose $\mathbf{C}$ is a locally small category. If for all objects $z \in \mathbf{C}$ we have $\mathbf{C}(z, x) \cong \mathbf{C}(z, y)$ (these need to be natural too) then $x \cong y$*

Proof. There is a natural isomorphism of representable functors $\mathbf{C}(-, x) \cong \mathbf{C}(-, y)$ induced by the collection of isomorphisms $\mathbf{C}(z, x) \cong \mathbf{C}(z, y)$. Since the Yoneda embedding is faithful $\mathcal{Y}(x) \cong \mathcal{Y}(y)$ implies that $x \cong y$. □

Remark 0.5. I believe this is where the idea of "the arrows are the ones that are important" comes into play, since we're saying that objects can be determined simply by looking at arrows into these objects.

I did not make it past the Yoneda lemma for this week