Chapter 7 Summary

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1 Naturality

The main thing in this chapter is the concept of naturality.

Definition 1.1. Suppose we had $F, G : \mathbf{C} \to \mathbf{D}$ functors. Then a **natural transformation** $\eta : F \to G$ is a collection of morphisms $\{\, \eta_c \,\}_{c \in \mathbf{C}}$ for each object $c \in \mathbf{C}$ such that the following commutes:

$$\begin{array}{ccc} Fc & \xrightarrow{\eta_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow[\eta_{c'}]{} & Gc' \end{array}$$

1.1 Functor categories

Now that we have natural transformations, functors from fixed categories actually forms a category, denoted $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$, where the morphisms are natural transformations. The identity natural transformation 1_F is a collection of morphisms $\{\, 1_F \,\}_{c \in \mathbf{C}}$ for each $c \in \mathbf{C}$ where $(1_F)_c = 1_{Fc}$, the identity morphism on the object Fc. Composition of natural transformations is given by $(\vartheta \circ \eta)_c = \vartheta_c \circ \eta_c$.

Example 1.2. Suppose we are in a category with products. Then $A \times B \cong B \times A$ naturally, by the natural transformation $t_{(A,B)} \circ \langle a, b \rangle = \langle b, a \rangle$. //

Example 1.3. Fix a field $\Bbbk$ and consider finite dimensional vector spaces V over $\Bbbk$. A classic example is that $V \cong V^{**}$ naturally. For each $V \in \mathrm{Vect}(\Bbbk)$, define η_V to be the map $v \mapsto (\epsilon_v : V^* \to \Bbbk)$ where ϵ is the evaluation map, that is given a linear functional $f \in V^*$, $f : V \to \Bbbk$, $\epsilon_v(f) = f(v)$. Now call the collection of all such η_V's η. Then η is a natural transformation from $1_{\mathbf{Vect}} \to (-)^{**}$. //

Importantly now since functors between fixed categories form a category we can see that **Cat** is a CCC.

> **Theorem 1.4.** **Cat** is cartesian closed, where $\mathbf{D}^{\mathbf{C}}$ is $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$

2 Monoidal categories

Joke 2.1. *A monad is a monoid in the category of endofunctors, what's the problem?*

An endofunctor is a functor where the domain and codomains are the same. Fix a category $\mathbf{C}$ and consider the category of endofunctors of $\mathbf{C}$. Now define $G \otimes F = G \circ F$ for endofunctors G, F. To define $\alpha \otimes \beta$, where $\alpha : G \to G'$, $\beta : F \to F'$ are natural transformations, let $c \in \mathbf{C}$ be an arbitrary object and define a morphism $GF(c) \to G'F'(c)$ by taking $G'(\beta_c) \circ \alpha_{F(c)}$ (or equivalently, $\alpha_{F'(c)} \circ G(\beta_c)$) The collection of such morphisms forms a natural transformation: $\alpha \otimes \beta : GF \to G'F'$.

3 Equivalence of categories

Equivalence of categories captures in the intuitive notion that some categories are "kind of the same".

Definition 3.1. Suppose we had categories $\mathbf{C}, \mathbf{D}$. Then $\mathbf{C} \simeq \mathbf{D}$ if we have natural isomorphisms $\alpha : 1_{\mathbf{C}} \to FE$ and $\beta : 1_{\mathbf{D}} \to EF$

A noteworthy property is that if F is part of an equivalence of categories then it's full and faithful. It's also essentially surjective on objects, meaning for every object in the codomain of F, there is a object in the domain such that applying F to this object is isomorphic to that object.