Week 6 Summary

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Honestly I did not understand much past cartesian closed categories...

Exponentials intuitively feel like a generalization of hom-sets. I believe the important thing here is that exponentials themselves are objects within the same category as the things you are taking exponentials over.

Definition 0.1. Suppose $\mathbf{C}$ has binary products. Let $B, C \in \mathbf{C}$ be objects. Then an exponential of B, C is an object denoted C^B and an arrow $\epsilon : C^B \times B \to C$ such that given any object A and a morphism $f : A \times B \to C$ there is a unique $\tilde{f} : A \to C^B$ such that $\epsilon(\tilde{f} \times 1_B) = f$.

We can view this intuitively as taking f, defining a new map $\tilde{f}$ which behaves as wrapping f into a new map and then for particular values $a \in A, b \in B$ we can evaluate $\tilde{f}$ at a. (The analogy is only valid for sets) This also gives us that $\hom_{\mathbf{C}}(A \times B, C) \cong \hom_{\mathbf{C}}(A, C^B)$.

Definition 0.2. A category is cartesian closed if it has all finite products and exponentials

The idea here is that if we have finite products, and the "hom sets" are also objects in the category then it is cartesian closed. (Not sure why it's called cartesian).

Example 0.3. We know that **Sets** is cartesian closed. First of all **Sets** is locally small so $\hom_{\mathbf{Sets}}(A, B) \in$ **Sets**. Also, sets has finite products. If we have $f : A \times B \to C$ then $\tilde{f} : A \to C^B$ can be defined as such: for all $a \in A$, $\tilde{f}(a) = g_a$, $g_a : B \to C$ where $g(b) = f(a, b)$. Then $\epsilon(\tilde{f}(a), b) = g_a(b)$ where ϵ simply evaluates $\tilde{f}(a)$ at b. //

Example 0.4. Another category that is Cartesian closed is **Cat**, since we can take $\mathrm{Fun}(C, D)$, the category of functors between C and D. The proof of this is kind of long though. //