1 Pullbacks

Definition 1.1. Suppose we had morphisms $f : A \to C$ and $g : B \to C$. Then the pullback of f and g is a pair of arrows $p_1 : P \to A$ and $p_2 : P \to B$ such that $fp_1 = gp_2$, and given any $z_1 : Z \to A$ and $z_2 : Z \to B$ such that $z_1 = p_1 u, z_2 = p_2 u$, there is a unique $u : Z \to P$ such that $z_1 = p_1 u$ and $z_2 = p_2 u$

Pullbacks are unique, so we can denote the pullback of C as $A \times_C B$

Proposition 1.2. *Suppose C has products and equalizers. Suppose we had morphisms $f : A \to C$ and $g : B \to C$. Then if $A \times B$ is the product of A and B, and E is an equalizer of $f\pi_1$ and $g\pi_2$, where $p_1 = \pi_1 e, p_2 = \pi_2 e$, p_1, p_2 is a pullback of f, g.*

Lemma 1.3 (Two pullback lemma)**.** *This diagram takes a long time to draw. Basically, if the 2 squares are pullbacks, the outer rectangle is a pullback. If the right square and outer rectangle are pullbacks, then the left square is too*

Corollary 1.4. *The pullback of a commutative triangle is a commutative triangle*

2 Limits

Definition 2.1. Let J, C be categories. Then a diagram of type J in C is a functor $D : J \to C$.

Definition 2.2 (Objects in the cone category)**.** A cone to a diagram D is an object, $C \in C$ and a collection of morphisms, $\{ c_j \}$, $c_j : C \to D_j$ such that for every $\alpha : i \to j$ in J, $D_\alpha c_i = c_j$.

Definition 2.3 (Morphisms in the cone category)**.** A morphism of cones $\vartheta : (C, \{ c_j \}) \to (C', \{ c'_j \})$ is a morphism $\overline{\vartheta} : C \to C'$ in C such that for every $j \in J$ we have $c_j = c'_j \overline{\vartheta}$.

Definition 2.4. If $D : J \to C$ is a diagram, then a limit for D is a terminal object in the category of cones to D. If J is finite then the limit is called a finite limit.

The limit object would be denoted $\varprojlim_j D_j$. It of course comes with a family of morphisms $\{ p_i \}$ such that $p_i : \varprojlim_j D_j \to D_i$. This object has the property that for any cone $(C, \{ c_j \})$ to D, there is a unique $u : C \to \varprojlim_j D_j$ such that for every $j \in J$ we have $p_j \circ u = c_j$.

We can now view products as a limit. Let $\mathbf{J}$ be the discrete category with 2 objects, 2 morphisms (which both have to be identities). Then $D : \mathbf{J} \to \mathbf{C}$ is a pair of objects $D_1, D_2 \in \mathbf{C}$. A cone of D is a object $C \in \mathbf{C}$ together with morphisms $c_i : C \to D_i$. A limit of D would be a terminal cone, but this exactly coincides with the product.

Now we can construct equalizers with limits. Let $\mathbf{J}$ be the category with 2 objects, $1, 2$ and morphisms $\alpha, \beta : 1 \to 2$ (of course the objects would need identity morphisms too but ignore those). Then a diagram of type $\mathbf{J}$ would be 2 objects: D_1, D_2 and morphisms $D_\alpha, D_\beta : D_1 \to D_2$. A cone would be $c_i : C \to D_i$ such that $D_\alpha c_1 = c_2 = D_\beta c_1$, so a limit for D would be an equalizer of D_α, D_β.

3 Continuity

Definition 3.1. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Then F preserves limits of type J if given a diagram $D : \mathbf{J} \to \mathbf{C}$ and a limit $p_j : L \to D_j$ then the cone $F(p_j) : F(L) \to F(D_j)$ is a limit for the diagram $F(D) : \mathbf{J} \to \mathbf{D}$. If F preserves all limits, it is continuous.

Proposition 3.2. *Let C be a locally small category. Then the representable functors* $\operatorname{Hom}(C, -)$ *is continuous.*

Weird that this definition is being introduced now but not in duality but:

Definition 3.3. Let $\mathbf{C}, \mathbf{D}$ be categories. Then $F : \mathbf{C}^{\text{op}} \to \mathbf{D}$ is a contravariant functor on $\mathbf{C}$, where if $f : A \to B$ is a morphism in $\mathbf{C}$ then it is mapped to $F(f) : F(B) \to F(A)$, and $F(g \circ f) = F(f) \circ F(g)$

Honestly I did not really understand the colimit part

Definition 3.4. A pushout is a pullback where you flip all the arrows and that's what it is and I don't want to draw the commutative diagram because it takes too long

Example 3.5. Let S^1, D^2 be as defined in topology. Then S^2 is the pushout of 2 of the same inclusion map $i : S^1 \to D^2$. //