1 Duality

If we have a proposition in the language of category theory, we may substitute $g \circ f$ for $f \circ g$, codomain for domain, and domain for codomain. This is the dual of the proposition. Since the axioms of category theory are the dual of themselves, any statement that can be deduced from the axioms means that the dual statement can also be deduced. This is what "flipping of the arrows" is doing.

If a statement holds in all categories, the dual statement also holds.

Example 1.1. The dual of an isomorphism is an isomorphism. //

Example 1.2. The monics are the epics in the dual category. The epics are the monics in the dual category. //

Example 1.3. The dual of an initial object is a terminal object. //

2 Coproducts

Definition 2.1. A coproduct is the dual of a product. Indeed, this is just the

$$\begin{tikzcd}
A \arrow[r, "q_1"] \arrow[dr, "z_1"'] & Q \arrow[d, dotted] & B \arrow[l, dotted, "q_2"'] \arrow[dl, "z_2"] \\
& Z &
\end{tikzcd}$$

UMP of products, but with all the arrows reversed.

Coproducts are more enlightening when considering examples

Example 2.2. In the category of topological spaces, the coproduct topology corresponds exactly to the disjoint union topology. See the above figure. (Replace q_1, q_2 with the canonical injection when dealing with coproducts, and canonical projections when dealing with products). //

3 Equalizers

Definition 3.1. If we have morphisms $f, g : A \to B$, then an equalizer of f, g is an object E and an arrow $e : E \to A$ such that the following diagram commutes: In particular, given any morphism $z : Z \to A$ such that $fz = gz$ then there

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \\
Z \arrow[u, dotted, "u"] \arrow[ur, "z"'] & &
\end{tikzcd}$$

exists a unique u such that $z = eu$

$$
\begin{tikzcd}
\operatorname{Ker}\phi \arrow[r, hook, "e"] & G \arrow[r, shift left, "\phi"] \arrow[r, shift right, "0"'] & H \\
K \arrow[u, hook, "i"] \arrow[ru, "k"'] & &
\end{tikzcd}
$$

Example 3.2 (Not stolen from the textbook)**.** Let G, H be groups and ϕ be a group homomorphism. Let 0 denote the zero homomorphism. Then $\ker \phi$ is an equalizer of $\phi, 0$, and e is the canonical inclusion. (Since the kernel is a subgroup). It turns out that if K is any group such that $\phi k = 0k$ then it must be that K is a subgroup of $\ker \phi$ //

Proposition 3.3. *If E is an equalizer of f, g and e is the associated map then e is monic*

Proof. By definition the uniqueness of the map into E guarantees this. □

4 Coequalizers

Definition 4.1. Coequalizers are the dual of equalizers.

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[rd, "z"'] & Q \arrow[d, dashed, "u"] \\
 & & Z
\end{tikzcd}
$$

A decent example of coequalizers is equivalence relations. If you have an equivalence relation R defined on X, then for any $f : X \to Y$ there exists a function $\overline{f}$ given by the map $[x] \mapsto f(x)$. We have that $\overline{f}\pi = f$. π is the coequalizer of r_1, r_2 where $r_1, r_2 : R \to X$. $\overline{f}$ is well defined.

Remark 4.2. The definition of $\overline{f}$ here is very suggestive of the first isomorphism theorem from abstract algebra.

By duality we have that the associated map of a coequalizer is monic.