Chapter 10 Summary

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March 2024

1 Equational adjoints

Given functors and natural transformations from identities we can define adjoints equationally. These are called the triangle identities.

Suppose $F \dashv U$, where $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$. Let η, ε be the unit and counits.

For every $d \in \mathcal{D}$ we have $U(d) = U(\varepsilon_d) \circ \eta_{U(d)}$, and for every $c \in \mathcal{C}$ we have $F(c) = \varepsilon_{F(c)} \circ F(\eta_c)$. Treating $U\varepsilon$, ε_F, η_U and $F\eta$ as natural transformations, we have $U\varepsilon \circ \eta_U = 1_U$, and $\varepsilon_F \circ F\eta = 1_F$. These two equalities are called the triangle identities, and if these hold, $F \dashv U$ with unit and counit η, ε. (The converse is true, but follows by definition)

2 Monads from Adjunctions

Now we shall see that adjunctions give rise to monads. Suppose we have $F \dashv U$. Let $T = U \circ F$. This appears to be the endofunctor $\mathcal{C} \to \mathcal{C}$ that we shall make into our monad. Intuitively, the unit should be the unit $\eta : 1_{\mathcal{C}} \to T$. For the multiplication, consider $\varepsilon_{F(c)} : FUF(c) \to F(c)$. Applying U to this we have $U \circ \varepsilon_{F(c)} : UFUF(c) \to UF(c)$. But this is just $U \circ \varepsilon_{F(c)} : T^2(c) \to T(c)$, thus we can let $\mu : T^2 \to T$ be such that $\mu_c = U(\varepsilon_{F(c)})$. So we have a monad (T, η, μ).

Example 2.1. Let $F : \mathbf{Sets} \to \mathbf{Mod}_R$ and $U : \mathbf{Mod}_R \to \mathbf{Sets}$, where U is the forgetful functor from (left) R-modules and F is the free functor, sending a set to the free (left) R-module on it. Then $T = U \circ F$ is a monad, which is called the free R-module monad. In particular, $T(A)$ is the set of finite formal linear combinations of elements of A. The unit η has each component η_A which takes $a \in A$ to the singleton formal a linear combination (in particular it goes to $\chi_a : A \to R$ which sents a to the unit in R and assigns 0 to everything else). The multiplication natural transformations behaves as follows on each component μ_A: Given a formal sum of formal sums, distribute the coefficients. //

Example 2.2. Let $F : \mathbf{Sets} \to \mathbf{Grp}$ and $U : \mathbf{Grp} \to \mathbf{Sets}$ be the free and forgetful functors respectively, so we have $F \dashv U$. Let $T = U \circ F$. T is called the free group monad, which takes a set A to the set of finite words with letters in A, and their formal inverses. //

3 Adjunctions from monads

It turns out that adjunctions come from monads. However, the intermediate category seems kind of contrived and perhaps does not reveal too much about the monad.