Chapter 1 Summary

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1 Important Definitions

In short categories have objects, morphisms, identity morphisms exist, you can compose morphisms and composition is associative and unital. To save space the definition of a category is not given.

Definition 1.1 (Isomorphisms)**.** Let C be a category, and $f : A \to B$ be a morphism in C. f is an **isomorphism** if there exists another morphism $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$.

Inverses are unique but the proof is left out from this summary to save space.

2 Constructions

We can construct more categories from categories.

Definition 2.1 (Product Category)**.** Let C, D be categories. Then one can take a product of categories denoted $C \times D$ where the objects are pairs (c, d) for $c \in \mathrm{Obj}(C)$ and $d \in \mathrm{Obj}(D)$. Morphisms in $C \times D$ are pairs $(f, g) : (c, d) \to (c', d')$ where $f \in \mathrm{Mor}(C)$ and $g \in \mathrm{Mor}(D)$. Composition and units are defined componentwise.

This definition can easily be extended to finite products, but I'm not too sure about infinite products.

Now we have the opposite category, a concept fundamental to duality. In short the opposite category has morphisms where the codomain and domain are swapped (as compared with the original category). In particular the opposite of a category contains the same data, but when we compose arrows in the opposite category we need to compose them in the original category.

Definition 2.2 (Opposite/Dual Category)**.** Let C be a category. Then we can define a new category C^{op} where the objects are the same as in C, but morphisms in C^{op} are morphisms in C where if $f : c \to c'$ is a morphism in C^{op} then it is a morphism in C, $f : c' \to c$. Composition of morphisms in C^{op}, $f \circ g$, is the composition of morphisms in C, $g \circ f$.

Free categories are categories generated by graphs. If G is a graph then we can consider the vertices of G as objects and the arrows as paths, where paths are simply sequences of edges which are finite in length. The domain of an arrow is defined to be the origin of the first edge in the sequence, and the codomain is defined to be the destination of the last edge in the sequence.

3 Transformations between Categories

We can transform between categories with functors. Functors in particular preserve the categorical structure, namely, the associativity and unital composition.

Definition 3.1 (Functor)**.** Let C, D be categories. A **functor** is a map $F : C \to D$ such that for each $c \in \mathrm{Obj}(C)$, $F(c) \in \mathrm{Obj}(D)$ and for each $f : c \to c' \in \mathrm{Mor}(C)$, $F(f)$ is a morphism with domain $F(c)$ and codomain $F(c')$. Also, $F(g \circ f) = F(g) \circ F(f)$. In addition, we have that $F(\mathrm{Id}_c) = \mathrm{Id}_{F(c)}$

Lemma 3.2. *Functors preserve isomorphisms.*

Of course one can think of categories and functors as being objects and morphisms, leading to the nice and cursed concept of categories of categories.

4 Size of categories

We have small, locally small and large categories. Small categories only have a set's worth of objects and morphisms. Locally small categories have a set's worth of morphisms between any 2 objects. Large categories are categories that are not locally small.