Abelian Categories

Robert

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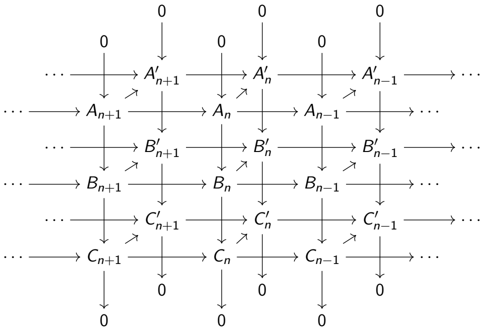

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Homological Algebra

Traditionally, we do homological algebra within categories such as **Ab** and $\mathbf{Mod}_R$ (this means left R-modules). Abelian categories are a generalization of what makes these categories so nice to do homological algebra in.

Definition

A category $\mathcal{C}$ is called *additive* if

1. $\mathcal{C}(a, b)$ is an abelian group, where composition distributes over addition.
2. There is an object that is both initial and terminal. We call this the *zero object*.
3. $\mathcal{C}$ has binary products (and thus finite ones)

Definition

A category $\mathcal{A}$ is called *abelian* if

1. It is an additive category.
2. Every morphism has a kernel and a cokernel
3. Every monomorphism is a kernel, every epimorphism is a cokernel

The kernel of f is defined to be the equalizer of f and 0. The zero morphism $0 : A \to B$ is obtained by taking the composition $A \to 0 \to B$.

Examples of abelian categories

- Category of abelian groups.

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- The category of chain complexes on an abelian category.

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- Category of left R-modules.
- The category of chain complexes on an abelian category.
- **LCA** is unfortunately not abelian.

Chain complexes

Definition

Let $\mathcal{A}$ be an abelian category. A *chain complex* in $\mathcal{A}$ is a sequence of objects (C_n) and a sequence of morphisms (∂_n) where $\partial_n : C_n \to C_{n-1}$ and it has the property that $\partial_{n+1} \circ \partial_n = 0$

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \qquad (C_{n+1} \xrightarrow{0} C_{n-1})$$

We denote this as (C_n, ∂).

Category of chain complexes

Chain complexes from an abelian category $\mathcal{A}$ form the objects in the category of chain complexes on $\mathcal{A}$. The morphisms in this category are sequences of morphisms (f_n) such that squares commute like so

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \longrightarrow & C'_n & \longrightarrow & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

and we write $(C_n, \partial) \to (C'_n, \partial')$.

Exact sequences

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

We say that a sequence (C_n, ∂) is *exact at* C_n if $\operatorname{im} \partial_{n+1} = \ker \partial_n$.
The image of a morphism $f : A \to B$ is defined to be $\ker(\operatorname{coker} f)$.

Short exact sequences

A *short exact sequence* is a sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

which is exact at A, B, C.

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- If f is both epi and monic, it is an isomorphism. Compare this with the category of abelian groups, and R-Mod.

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- Given any morphism f, we can factor it like $f = me$ where m is monic and e is epic.
- If f is both epi and monic, it is an isomorphism. Compare this with the category of abelian groups, and R-Mod.
- Given a short exact sequence $0 \to A \to B \to C \to 0$, if $f : A \to B$ and $g : B \to C$ then we have f monic and g epi.

Exact functors

Definition

A functor F is *left-exact* if the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C$ being exact implies that the sequence $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ is exact. A functor being *right-exact* is defined similarly.

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A functor that is both left and right exact is called exact. There are many equivalent definitions of exact functor.

Examples of exact functors

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- The covariant hom functor into abelian groups is left-exact. The contravariant one is right-exact.

Diagram lemmas

- Five lemma
- Snake lemma

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Given $x, y \in_m a$, define $x \sim y$ if and only if there are epis u, v such that $xu = yv$.

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Given $x, y \in_m a$, define $x \sim y$ if and only if there are epis u, v such that $xu = yv$.

The symbol $\in_m$ is chosen purposefully for intuition.

Abelian categories are intuitive

Theorem

Let $\in_m$ denote membership in an abelian category.

1. $f : a \to b$ *is monic if and only if for all* $x \in_m a$, $fx \sim 0$ *implies that* $x \sim 0$;
2. $f : a \to b$ *is monic if and only if for all* $x, x' \in_m a$, $fx \sim fx'$ *implies* $x \sim x'$;
3. $g : b \to c$ *is epi if and only if for every* $z \in_m c$ *there exists a* $y \in_m b$ *such that* $gy \sim z$;
4. $h : r \to s$ *is zero if and only if for all* $x \in_m r$, $hx \sim 0$;
5. *A sequence* $a \xrightarrow{f} b \xrightarrow{g} c$ *is exact at* b *iff* $gf = 0$ *and for every* $y \in_m b$ *such that* $gy \sim 0$ *there exists* $x \in_m a$ *so that* $fx \sim y$;
6. *Given* $g : b \to c$ *and* $x, y \in_m b$ *with* $gx \sim gy$, *there is some* $z \in_m b$ *such that* $gz \sim 0$; *and if any* $f : b \to d$ *is such that* $fx \sim 0$ *then we have* $fy \sim fz$, *additionally, if* $h : b \to a$ *is such that* $hy \sim 0$ *we have* $hx \sim -hz$.

Five Lemma

Lemma (Five lemma)

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{g_1} & A_2 & \xrightarrow{g_2} & A_3 & \xrightarrow{g_3} & A_4 & \xrightarrow{g_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow[h_1]{} & B_2 & \xrightarrow[h_2]{} & B_3 & \xrightarrow[h_3]{} & B_4 & \xrightarrow[h_4]{} & B_5
\end{array}$$

Suppose the rows are exact, and f_1, f_2, f_4, f_5 are isomorphisms. Then f_3 is an isomorphism.

Freyd-Mitchell

Theorem (Freyd-Mitchell Embedding Theorem)

Let $\mathcal{A}$ be a small Abelian category. Then there is a ring with unity R and a functor $F : \mathcal{A} \to \mathbf{Mod}_R$ (left R-module category) such that F is full, faithful and exact.

Snake lemma with Freyd-Mitchell

$$
\begin{array}{ccccccccc}
 & & F(\ker\alpha) & \longrightarrow & F(\ker\beta) & \longrightarrow & F(\ker\gamma) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Fa & \xrightarrow{Fm} & Fb & \xrightarrow{Fe} & Fc & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F\alpha} & & \downarrow{\scriptstyle F\beta} & & \downarrow{\scriptstyle F\gamma} & & \\
0 & \longrightarrow & Fa' & \xrightarrow{Fm'} & Fb' & \xrightarrow{Fe'} & Fc' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & F(\operatorname{coker}\alpha) & \longrightarrow & F(\operatorname{coker}\beta) & \longrightarrow & F(\operatorname{coker}\gamma) & &
\end{array}
$$