Abelian Categories

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Abstract

We aim to give a brief overview on the theory of abelian categories and draw a comparison with the usual tools of homological algebra, which is often studied in the categories **Ab** and $\mathbf{Mod}_R$. Additionally, we discuss the Freyd-Mitchell embedding theorem and some shortcomings of the theorem.

1 Introduction

Classically, homological algebra was done in categories such as **Ab** and R-Mod. Abelian categories are in a sense, the most general category in which homological algebra can be done within. The axioms of an abelian category give us just enough to do homological algebra in. With that, let's dive right in with some definitions.

Definition 1.1. An *additive category* is a category $\mathcal{C}$ such that

1. Given any 2 objects $A, B \in \text{obj}\,\mathcal{C}$, $\mathcal{C}(A, B)$ forms an abelian group.
   (a) Let $f, g : A \to B$. If $h : B \to C$ is some other morphism, then we have $h \circ (g + f) = h \circ g + h \circ f$
2. There is an object $0 \in \text{obj}\,\mathcal{C}$, which is both initial and terminal. This object is called the *zero object*.
3. Given any 2 objects A, B, the product of A and B is also in $\mathcal{C}$ (and thus by induction, we have finite products).

Note that when we say $\mathcal{C}(A, B)$ is an abelian group, the additive identity is the morphism $0 : A \to B$, which is obtained by factoring through the zero object: $A \to 0 \to B$.

Now another definition, which we will use pretty often.

Definition 1.2. The *kernel* of f, is the equalizer of f and 0.

It is easy to see that this definition is only valid in categories with a 0 morphism. Since the kernel is given by a universal property (equalizers), it is unique up to isomorphism. Cokernels are dually defined by taking the coequalizer of f and 0.

Finally we can define an abelian category.

Definition 1.3. An *abelian category* $\mathcal{C}$ is an additive category with additional properties:

1. Given any $f \in \text{mor}\,\mathcal{C}$, we can form $\ker f$.
2. Given any $f \in \text{mor}\,\mathcal{C}$, we can form $\text{coker}\, f$.
3. If $f \in \text{mor}\,\mathcal{C}$ is a monomorphism, it is the kernel of some morphism.
4. If $f \in \text{mor}\,\mathcal{C}$ is an epimorphism, it is the cokernel of some morphism.

Taking a look at some examples of abelian categories will provide some insight into why these axioms are formulated as such. Unsurprisingly, the categories in which we do homological algebra are abelian categories.

Example 1.4 (Category of (left) R-modules)**.** Let R be a ring (with unity). Let $\mathcal{C} = \mathbf{Mod}_R$, the category of (left) R-modules. Then $\mathcal{C}$ is an abelian category. It's not too hard to see that $\mathcal{C}$ is an additive category, where the abelian group structure of $\mathcal{C}(M, N)$ is provided by pointwise addition in N. Now kernels and cokernels are just the usual definition of kernels and cokernels of modules. //

Example 1.5 (The category **Ab**)**.** The category of abelian groups, **Ab** is an abelian category. Intuitively this makes sense because it's the category of abelian groups. But let's actually see why. Firstly **Ab** is additive: the abelian group structure $\text{Hom}(G, H)$ is inherited from H, we can form products, and we have the trivial group serving as the 0 object. From here, it's not too hard to see why the category is abelian. Let $f : G \to H$ be a group homomorphism. $\ker f$ is formed in the usual way: the set of elements that are mapped to 0 by f. $\text{coker}\, f$ is formed by taking $H/\,\text{im}\, f$. //

The following examples are constructions of abelian categories from other abelian categories.

Example 1.6 ($\text{Fun}(\mathcal{J}, \mathcal{C})$)**.** If $\mathcal{C}$ is a (small) abelian category, then $\text{Fun}(\mathcal{J}, \mathcal{C})$ is an abelian category too. This situation is very similar to how the group structure of G can be inherited by $\text{Hom}(S, G)$ if S is any set, by simply defining the operations pointwise. //

Example 1.7 (Category of chain complexes)**.** Given an abelian category $\mathcal{A}$, the category of chain complexes on $\mathcal{A}$ is also an abelian category. We define a *chain complex* to be a sequence of objects (C_n) and a sequence of morphisms (∂_n)

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0$. We write this as (C_n, ∂). These are objects in the category of chain complexes. A morphism in the category of chain complexes is a sequence of morphisms $f_n : C_n \to C'_n$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & C'_{n+1} & \longrightarrow & C'_n & \longrightarrow & C'_{n-1} & \longrightarrow & \cdots \end{array}$$

Consider chain complexes (C_n, ∂) and (C'_n, ∂'). If f, g are morphisms from C_n to C'_n, we have morphisms $f_n, g_n : C_n \to C'_n$ in $\mathcal{A}$, which can evidently be added together. Indeed, we can see that $\partial_n \circ (f_n + g_n) = \partial_n \circ f_n + \partial_n \circ g_n = f_{n+1} \circ \partial'_n + g_{n+1} \circ \partial'_n$ which is as desired. The other properties are also easy to verify. For example, the kernel of f in the category of chain complexes is a sequence of kernels $\ker f_n$ for each f_n in $\mathcal{A}$. //

Example 1.8 (Sheaves)**.** If X is a topological space, the category of all sheaves of abelian groups on X is an abelian category. Sheaves are not discussed in this report but we shall leave exposition about sheaves to Tamis. //

1.1 Exact sequences and homology

Let us begin by recalling the definition of an exact sequence with the more familiar category of abelian groups.

Definition 1.9. A sequence of abelian groups (G_n) is exact at G_k if $\operatorname{im} \partial_{k+1} = \ker \partial_k$.

$$\cdots \longrightarrow G_{k+1} \xrightarrow{\partial_{k+1}} G_k \xrightarrow{\partial_k} G_{k-1} \longrightarrow \cdots$$

A chain complex (C_n, ∂) is said to be exact if it is exact at every C_n.

An example of a chain complex is taking free abelian group with basis the singular n-simplexes on a topological space X.

For an abelian category, it is the same definition, only instead of abelian groups, we have objects in an abelian category. But first, we need to define what an image is. If $f : A \to B$ is a morphism, then we can define $\operatorname{im} f = \ker(\operatorname{coker} f)$.

Definition 1.10. A *short exact sequence* is a sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

which is exact at A, B, C.

Equivalently, we can say that a sequence is short exact if $f = \ker g$ and $g = \operatorname{coker} f$.

Now we can talk about homology. If $i : A \to B$ is a monomorphism, then we can define the quotient of B by A, denoted B/A as $\operatorname{coker} i$. This agrees with the usual definition of quotient in abelian groups and modules.

Definition 1.11. Given a chain complex (C_n, ∂), the *n-th homology object*, denoted $H_n(C)$, is defined as $\ker \partial_n / \operatorname{im} \partial_{n+1}$.

The idea of homology is that it measures how much a sequence deviates from being an exact sequence. Indeed, (C_n, ∂) is exact if and only if the nth homology is zero for every n. An example of an exact sequence is the chain complex taken on a point ([Hat02, Prop 2.8]).

We do not discuss homology much in the report, but it is included here for completeness.

2 Basic facts

It is helpful to see some properties of abelian categories. Most proofs are omitted since they are unsightful.
Proposition 2.1. *$\mathcal{C}$ is an abelian category if and only if $\mathcal{C}^{op}$ is an abelian category.*
Proposition 2.2. *If $\mathcal{A}$ is an abelian category and f is a morphism in $\mathcal{A}$, then f can be factored as $f = me$, where m is monic and e is epi. Moreover, $m = \ker(\operatorname{coker} f)$ and $e = \operatorname{coker}(\ker f)$.*
Proposition 2.3. *g is monic if and only if $g = \ker(\operatorname{coker} g)$. Dually, g is epi if and only if $g = \operatorname{coker}(\ker g)$*
Proposition 2.4 ([Mac78])**.** *In an abelian category, if f is epi and monic, then f is an isomorphism*
Proposition 2.5. *In an abelian category, given the pullback of f and g, if f is epi, then so is f'*

$$\begin{array}{ccc} b \times_c d & \xrightarrow{f'} & d \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ b & \xrightarrow[f]{} & c \end{array}$$

Now, a few properties about exact sequences in abelian categories.
Proposition 2.6. *If*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is short exact, then f is a monomorphism and g is an epimorphism.
Proposition 2.7. *Given the pullback square as in Proposition 2.5, k factors through the kernel of f', denoted k', such that $k = g'k'$.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & a & \overset{k'}{\dashrightarrow} & b \times_c d & \xrightarrow{f'} & d \\ & & \| & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & a & \xrightarrow{k} & b & \xrightarrow[f]{} & c \end{array}$$

Of course, we are all pretty familiar with sets and taking elements of them to do proofs. The following theorem tells us that our intuition from sets carries over, and justifies the use of the $\in_m$ notation for membership. But first a definition.

Definition 2.8. If x is a morphism with codomain a, then x is a *member of* a, denoted $x \in_m a$. Given $x, y \in_m a$, define $x \sim y$ if and only if there are epis u, v such that $xu = yv$.

The relation $\sim$ as defined is an equivalence relation, but the proof of this fact is omitted as it is not insightful (to prove the transitivity of $\sim$ we use Proposition 2.5, see [Mac78, p. 204])

Theorem 2.9. Let $\in_m$ denote membership in an abelian category.

1. $f : a \to b$ is monic if and only if for all $x \in_m a$, $fx \sim 0$ implies that $x \sim 0$;
2. $f : a \to b$ is monic if and only if for all $x, x' \in_m a$, $fx \sim fx'$ implies $x \sim x'$;
3. $g : b \to c$ is epi if and only if for every $z \in_m c$ there exists a $y \in_m b$ such that $gy \sim z$;
4. $h : r \to s$ is zero if and only if for all $x \in_m r$, $hx \sim 0$;
5. A sequence $a \xrightarrow{f} b \xrightarrow{g} c$ is exact at b iff $gf = 0$ and for every $y \in_m b$ such that $gy \sim 0$ there exists $x \in_m a$ so that $fx \sim y$;
6. Given $g : b \to c$ and $x, y \in_m b$ with $gx \sim gy$, there is some $z \in_m b$ such that $gz \sim 0$; and if any $f : b \to d$ is such that $fx \sim 0$ then we have $fy \sim fz$, additionally, if $h : b \to a$ is such that $hy \sim 0$ we have $hx \sim -hz$.

Properties 1-3 are extremely similar to the usual definition of injectivity and surjectivity with sets and functions. Property 4 is reminiscent of how we can show that a function is the zero map by simply checking that it evaluates to zero on every single element. Property 5 is comparable with the "usual" way exactness behaves in concrete categories. In concrete categories $g \circ f = 0$ immediately implies that $\operatorname{im} f \subseteq \ker g$, and for exactness we would have the following property: given any $y \in \ker g = \operatorname{im} f$, there is some $x \in a$ such that $f(x) = y$. The last property is meant to replace subtraction in abelian groups [Mac78].

Proof. Properties 1 and 2 follow immediately from the definition of monic. For property 3, take the pullback of g and z. Then we have the following situation:

$$\begin{array}{ccc} s & \xrightarrow{g'} & \bullet \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle z} \\ b & \underset{g}{\twoheadrightarrow} & c \end{array}$$

Since g is epi, g' is epi by Proposition 2.5 and thus $gy1_s = zg'$, so we have $gy \sim z$. Property 4 follows immediately. Properties 5 and 6 are omitted. ☐

3 Classic diagram lemmas

When doing homological algebra, there are a few diagram lemmas. The more well known ones include the five lemma, and the snake lemma. These lemmas hold in **Ab** and $\mathbf{Mod}_R$, which one can find proofs of in almost any textbook on homological algebra. We shall give a proof of these lemmas in an arbitrary abelian category.

Lemma 3.1 (Five lemma)**.**

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{g_1} & A_2 & \xrightarrow{g_2} & A_3 & \xrightarrow{g_3} & A_4 & \xrightarrow{g_4} & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \xrightarrow[h_1]{} & B_2 & \xrightarrow[h_2]{} & B_3 & \xrightarrow[h_3]{} & B_4 & \xrightarrow[h_4]{} & B_5 \end{array}$$

Suppose the rows are exact, and f_1, f_2, f_4, f_5 are isomorphisms. Then f_3 is an isomorphism.

The hypotheses can be weakened slightly. Indeed, we only need f_1 epi and f_5 monic instead of them being isomorphisms.

Now let us see a proof using the methods of Theorem 2.9.

Proof. We shall prove that f_3 is monic by using Theorem 2.9 part 1 (and apply duality to get that it is epi). Pick $x \in_m A_3$, such that $f_3 x \sim 0$. Then $f_4 g_3 x = h_3 f_3 x \sim h_3 0 = 0$ so $f_4 g_3 x \sim 0$. Since f_4 is monic we have $g_3 x \sim 0$. Since we are exact at A_3, apply part 5 of Theorem 2.9, there is some $y \in_m A_2$ such that $g_2 y \sim x$. Then $0 \sim f_3 x \sim f_3 g_2 y \sim h_2 f_2 y$. By exactness at B_2 and the fact that $f_2 y \in_m B_2$ there is some $y' \in_m B_1$ such that $h_1 y' \sim f_2 y$. Since f_1 is epi, there is some $z \in_m A_1$ such that $f_1 z \sim y'$ and so we have $h_1 f_1 z \sim f_2 y$. Since $h_1 f_1 z = f_2 g_1 z$ we have $f_2 g_1 z \sim f_2 y$. Since f_2 is monic we have $g_1 z \sim y$. Since we have shown above that $x \sim g_2 y$, and $g_2 y \sim g_2 g_1 z \sim 0$ as $g_2 g_1 = 0$. This shows x is zero. Now since x is arbitrary, by Theorem 2.9 part 1 we see that f_3 is monic. ☐

Notice how the proof is very similar to doing it in the category of abelian groups, by picking some element in the kernel and chasing it around.

Lemma 3.2 (Snake lemma)**.** *Consider*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

where the rows are exact, and all the squares commute. Then, there exists an exact sequence

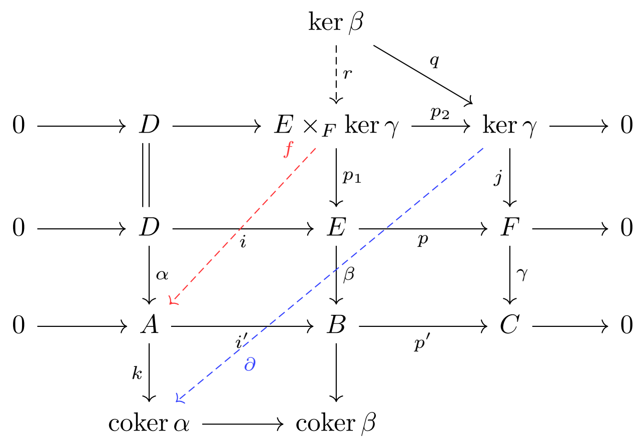
$$0 \longrightarrow \ker \alpha \longrightarrow \ker \beta \longrightarrow \ker \gamma \longrightarrow \operatorname{coker} \alpha \longrightarrow \operatorname{coker} \beta \longrightarrow \operatorname{coker} \gamma \longrightarrow 0$$

Proof Sketch. By universal property of kernels and cokernels, we already have

$$0 \longrightarrow \ker \alpha \longrightarrow \ker \beta \longrightarrow \ker \gamma$$

$$\operatorname{coker} \alpha \longrightarrow \operatorname{coker} \beta \longrightarrow \operatorname{coker} \gamma \longrightarrow 0$$

The morphisms between the kernels and cokernels are unique and the sequence is exact. The hard part of this proof is finding the connecting morphism $\ker \gamma \to \operatorname{coker} \alpha$. We do so by examining the following diagram [Cre21].



The sequence $0 \to D \to E \times_F \ker \gamma \to \ker \gamma \to 0$ is exact by Proposition 2.7. Now, $p'\beta p_1 = \gamma j p_2 = 0$, since $\gamma j = 0$. So there is a unique f (in red) such that $i'f = \beta p_1$. Since the top row is exact, $D \to E \times_F \ker \gamma$ is the kernel of p_2. Additionally, $k\alpha = 0$, so we may define ∂ to be the unique morphism such that $\partial p_2 = kf$. This is what we are looking for. The only thing left to do is to verify the exactness of the sequence found. □

4 Transformations between abelian categories

A functor that preserves the abelian category structure is called an exact functor. From the name, we can guess that an exact functor is a functor that preserves exact sequences. There are multiple equivalent definitions of exact functors, this one is from [Mac78].

Definition 4.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A functor $F : \mathcal{A} \to \mathcal{B}$ is *exact* if it preserves finite limits and colimits.

Now we can have a look at some functors which are left exact or exact.

Example 4.2. If $\mathcal{A}$ is equivalent to $\mathcal{B}$ (both are abelian categories), then the functors connecting them are exact. //

Example 4.3 (Representable functors)**.** Given an abelian category $\mathcal{A}$ and $a \in \operatorname{obj} \mathcal{A}$, $\operatorname{Hom}(a, -) : \mathcal{A} \to \mathbf{Ab}$ is a (covariant) left-exact functor. a is projective if and only if $\operatorname{Hom}(a, -)$ is exact [Jac09, Prop 3.9]. The contravariant version of that functor is right-exact and is exact if and only if a is injective (by duality). //

Example 4.4. In the category of vector spaces over a fixed field, the dual functor is a contravariant exact functor. //

We will also quickly define what it means for functors to be left and right exact.

Definition 4.5. A functor F is *left-exact* if the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C$ being exact implies that the sequence $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ is exact. A functor being *right-exact* is defined similarly.

A left exact functor preserves kernels, and a right exact functor preserves cokernels.

Proposition 4.6. *The following are equivalent:*

1. *F is exact*
2. $\ker(Ff) = F(\ker f)$ *and* $\operatorname{coker}(Ff) = F(\operatorname{coker} f)$
3. *F is additive and preserves kernels and cokernels.*
4. *F is left and right exact*

For a proof of these equivalences, see [Mac78] and [Vak17].

5 Embedding theorem

A discussion about abelian categories would not be complete without this embedding theorem. Essentially, this theorem tells us that we can prove facts about abelian categories in $\mathbf{Mod}_R$ and then use those to recover the desired morphisms. This theorem is somewhat useful and some authors quite like this. For instance, [Vak17] recommends proving diagram lemmas in the category of R modules first and applying the embedding theorem.

Theorem 5.1 (Freyd-Mitchell Embedding Theorem)**.** Let $\mathcal{A}$ be a small abelian category. Then, there is a ring with unity R and a functor $F : \mathcal{A} \to \mathbf{Mod}_R$ such that F is full, faithful and exact.

This theorem is nice because it saves some work: One can simply prove results in Mod_R and apply the Embedding theorem to translate it. As an example, let's see how this works by proving the snake lemma using the embedding theorem.

Proof of Lemma 3.2 by embedding theorem. By the embedding theorem, we have a ring with unity R and a fully faithful exact functor $F : \mathcal{A} \to \mathbf{Mod}_R$. Now, take F and apply it to our exact sequences[1]

$$\begin{array}{ccccccccc}
 & & F(\ker\alpha) & \longrightarrow & F(\ker\beta) & \longrightarrow & F(\ker\gamma) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Fa & \xrightarrow{Fm} & Fb & \xrightarrow{Fe} & Fc & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F\alpha} & & \downarrow{\scriptstyle F\beta} & & \downarrow{\scriptstyle F\gamma} & & \\
0 & \longrightarrow & Fa' & \xrightarrow{Fm'} & Fb' & \xrightarrow{Fe'} & Fc' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & F(\operatorname{coker}\alpha) & \longrightarrow & F(\operatorname{coker}\beta) & \longrightarrow & F(\operatorname{coker}\gamma) & &
\end{array}$$

The above diagram is in the category $\mathbf{Mod}_R$ and since F is exact, the rows are all exact. Thus the snake lemma applies and we have a homomorphism $\partial : F(\ker\gamma) \to F(\operatorname{coker}\alpha)$. Since F is fully faithful, there is a unique morphism $\tilde{\partial}$ from $\ker\gamma$ to $\operatorname{coker}\alpha$ such that $F(\tilde{\partial}) = \partial$. This completes the proof. □

However this theorem is not all flowers and roses. Firstly, this theorem doesn't preserve projective and injective objects. In particular, taking the category of finitely generated $\mathbb{Z}$-modules (abelian groups) as given here in [zcn14].

Secondly, it really only works on small abelian categories (as part of the hypothesis).

Finally, we also established in Section 3 that we can prove some of our diagram lemmas without the embedding theorem. In fact all of the diagram lemmas can be proven without appealing to the embedding theorem, and there aren't any well known results that rely on the embedding theorem [Bra13]. Using Theorem 2.9 we can adapt proofs in the category of abelian groups or R-Mod to a general abelian category.

[1] We first obtain the morphisms between the kernels and cokernels within the abelian category itself. It is possible to just get these in the category we embedded into as well, but I find that the proof this way is slightly cleaner.

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