An introduction to Simplicial Homology

A brief overview

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Table of Contents

What is Homology

Homology is a way to associate sequences of algebraic objects with other mathematical objects. In this presentation, we will only concern ourselves with associating abelian groups to topological spaces.

Motivation

The motivation for homology comes from being able to tell topological spaces from each other. Imagine a cup and a donut. How do we know these topological spaces are not the same as each other? Intuitively, a cup has no holes, but a donut has a hole. Homology gives us a rigorous way to identify holes in a topological space.

Notation

We will heavily abuse notation, and denote the trivial group and trivial homomorphism with 0.
Note that the equality sign will mean both equality and isomorphism. For example, $\langle 2 \rangle = \mathbb{Z}$

Simplicial Complexes

Simplicial complexes are a generalization of triangles. We'll denote an n simplex with the following notation:

$$[v_0, v_1, \ldots, v_n]$$

Where v_i are vectors in Euclidean space. Note that the ordering of the vertices does matter. In particular, if $i < j$ then $[v_i, v_j]$ is an edge where you go from v_i to v_j. $-[v_i, v_j]$ means you go from v_j to v_i. An example will be given in the next slide.

In this case, the direction $[v_0, v_1]$ is depicted by that arrow in the figure. Imagine you are an ant. So you start from v_0 and walk to v_1. The direction of $[v_1, v_2]$ is also depicted by the arrow in the figure. Likewise with $[v_0, v_2]$.

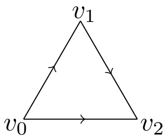


Figure: A standard 2-simplex

Standard n-simplex

A standard 3-simplex is a tetrahedron (triangular pyramid). We now give the definition of the standard n-simplex, Δ^n, which is simply the collection of unit $n+1$ vectors in $\mathbb{R}^{n+1}$.

$$\Delta^n = \{ v = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid ||v|| = 1, t_i \geq 0 \}$$

Of course, it is intuitively clear that the standard n-simplex is homeomorphic to any other n-simplex. We won't prove this here, but you can see [Hat02] for a proof.

Face of simplex

A face of an n-simplex is just an $n-1$-simplex, where you just delete one of the vertices. If the ith vertex is deleted we denote it like

$$[v_0, \dots, \hat{v}_i, \dots, v_n]$$

For example, if we write $[v_0, \hat{v}_1, v_2]$, it is the same as writing $[v_0, v_2]$. Intuitively, if we delete a vertex from a standard 2-simplex, we get a line. This makes much more intuitive sense when you consider that the faces of a standard 3-simplex are 2-simplexes, which are triangles. Indeed, triangles make up the faces of a tetrahedron.

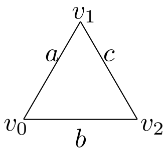


Figure: Faces of 2-simplex

In this picture, we can see that the faces of our 2-simplex are $[\hat{v}_0, v_1, v_2]$, $[v_0, \hat{v}_1, v_2]$ and $[v_0, v_1, \hat{v}_2]$ which correspond to c, b, a respectively, in that order.

Δ complexes

Let X be a Δ-complex. Intuitively, Δ-complexes are just spaces made unions of standard n-simplexes, where you identify certain faces together.
Formally, what this means is that X can be constructed as the quotient space of a disjoint union of n-simplexes Δ^n_α, with maps $\sigma_\alpha : \Delta^n \to X$ which identify Δ^n with each Δ^n_α. A good overview of quotient spaces has been covered in excellent presentation given by Mark on Friday Nov 24 at 3pm.

Free groups

Let S be a set of symbols. The free group on S, denoted F_S is the set of all finite length strings of characters on S and the inverses. For example, if $S = \{ a, b \}$, then $aba^{-1}b^{-1}$ is an element of S. If a character occurs with its inverse, it is the empty string. For instance, $aa^{-1}b = ab$.
If it is a free abelian group, $ab = ba$. This is much simpler to deal with. So, if $S = \{ a_1, \ldots, a_n \}$ then

$$\mathrm{FAb}(S) = \{ k_1 a_1 + \cdots + k_n a_n \mid k_i \in \mathbb{Z} \}$$

Basically the free abelian group on S is just all the linear combinations of the elements of S.

Define C_n to be the free abelian group with basis the n-simplexes of X.[1].

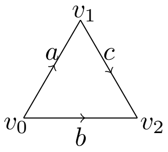


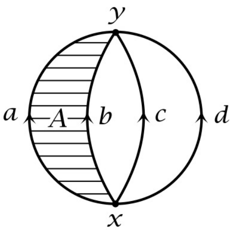
Figure: Δ-complex made from 3 1-simplexes

For example, consider this Δ-complex made of 3 1-simplexes (lines). Then, $C_0 = \text{FAb}(v_0, v_1, v_2), C_1 = \text{FAb}(a, b, c)$ and C_2 and above are all trivial since there are no 2-simplexes and so on.

[1]We are abusing notation from singular homology here, because it is easier to typeset. Hatcher actually calls this $\Delta_n(X)$

Motivation of homology groups

To allow our homology groups to detect holes, we'll consider quotient groups formed by cycles modulo boundaries. We may employ intuition from the 2 dimensional setting where a cycle is just following edges starting and ending at the same point. If the cycle is a boundary, there is no hole, so we have moduloed it away. For example, $a - b$ is a boundary of A, but $b - c$, $c - d$ and $b - d$ are all cycles that aren't boundaries.



Boundary Homomorphism

Let X be a Δ-complex. We can talk about the boundary of an n simplex in X, Δ^n_α, which is given by traversing the faces of Δ^n_α in a certain order.

Let $[v_0, v_1, v_2, v_3]$ be the standard 3-simplex. If we consider its boundary, it is given by

$$\begin{aligned}\partial(\Delta^3) &= [\hat{v_0}, v_1, v_2, v_3] \\ &- [v_0, \hat{v_1}, v_2, v_3] \\ &+ [v_0, v_1, \hat{v_2}, v_3] \\ &- [v_0, v_1, v_2, \hat{v_3}]\end{aligned}$$

Motivation of boundary homomorphism

So intuitively we captured the boundary of Δ^3 in $[v_0, v_1, v_2, v_3]$ with $\partial(\Delta^3)$. In particular, the boundary of a standard 3-simplex are the 4 faces that bounds it (intuitively). An easier object to visualize is the standard 2-simplex. We will see this in the next slide in the presentation.

Example of boundary homomorphism

Let A be the standard 2-simplex. (Think about A as the area bounded by the triangle thingy) The boundary of the standard 2-simplex is given by $\partial(A) = [\hat{v_0}, v_1, v_2] - [v_0, \hat{v_1}, v_2] + [v_0, v_1, \hat{v_2}]$. Notice that $-[v_0, \hat{v_1}, v_2]$ would be going from v_2 to v_0, because of the negative sign (the arrow in the drawing is not the correct direction, because I suck at TikZ).

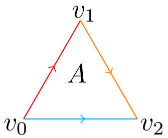


Figure: A standard 2-simplex with faces marked

Homology groups

Now we can finally define our homology groups. Recall that C_n is free abelian with basis of the n simplexes of X. Denote ∂_n to be a boundary homomorphism from Δ^n into our X.

$$\dots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$$

The n-th homology group is defined by

$$H_n(X) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$$

For example, intutively, $\operatorname{Ker} \partial_1$ captures all the linear combinations of 1-simplexes such that if you traverse them you start and end at the same point, the cycles. $\operatorname{Im} \partial_2$ captures all the boundaries of 2-simplexes.

Calculation of $H_1(S^1)$

We will now calculate the first homology group of S^1

We first construct S^1 like so:

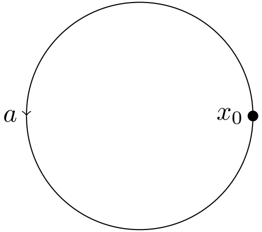


Figure: S^1 as a simplicial complex

It is clear from the figure that S^1 has a single 0-cell x_0 and a single 1-cell a.

By observation, we get that $C_0 = \text{FAb}(x_0)$, and that $C_1 = \text{FAb}(a)$. It is clear that for $i > 1$, $C_i = 0$, the trivial group. So,

$$\ldots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$$

Now, $\partial_1(a) = 0$, so $\text{Ker}\,\partial_1 = C_1$. And $\partial_2 = 0$, so it must be that $\Im\partial_2 = 0$. Thus $H_1(S^1) = \text{Ker}\,\partial_1 / \text{Im}\,\partial_2 = C_1 \approx \mathbb{Z}$

Remark

It is interesting to note that for a path-connected space X, $H_1(X)$ is the Abelianization of $\pi_1(X)$. See Theorem 2A.1 in [Hat02] for a proof

Extra Topics and Appendix

This is the end of the presentation. We do have some extra topics thoughx

Calculation of $H_1(S^n)$ for $n > 1$

We can construct S^n as follows. Let x_0 be a 0-cell (a point). Now, attach an $n - cell$, e^n by identifying the boundary to x_0. That is declare all the elements in ∂e^n to be equivalent to x_0. Now, notice that $C_0 = \text{FAb}(x_0)$ still, but $C_1 = 0$ is trivial. So $\operatorname{Ker} \partial_1 = 0$ and thus $H_1(S^n) = 0$

Brouwer Fixed Point Theorem

We will prove the Brouwer fixed point theorem in $n > 2$ using homology.

Theorem

Let $f : D^n \to D^n$ be a continuous map. Then, f has a fixed point, that is there is some $y \in D^n$ such that $f(y) = y$.

Proof.

Suppose for contradiction $f : D^n \to D^n$ is a continuous function with no fixed point. Define a retraction like we did in the last time. This retraction induces an injective group homomorphism from $H_{n-1}(D^n)$ to $H_{n-1}(S^{n-1})$. However, $H_{n-1}(D^n)$ is trivial, but $H_{n-1}(S^{n-1}) \approx \mathbb{Z}$. This contradiction completes the proof. □

Calculation of $H_n(S^n)$ for $n > 2$

Recall that S^n is made from a 0-cell and an n-cell. Clearly C_n is free abelian on one generator, C_{n-1} is trivial, C_{n+1} is also trivial. So $\operatorname{Ker} \partial_n = \mathbb{Z}$, $\operatorname{Im} \partial_{n+1} = 0$, so $H_n(S^n) = \mathbb{Z}$

References

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge New York: Cambridge University Press, 2002. ISBN: 9780521791601.