We have previously constructed the direct product of 2 groups. This enables us to build a bigger group using 2 smaller groups. However, many groups cannot be realized as a direct product of 2 groups, even the finite groups. If we relax the conditions on the product, it turns out we can do so.

The main "issue" that occurs with the direct product construction is if we had groups H, K, then both H, K (identified as subgroups of $H \times K$) appear as normal subgroups in the direct product. It seems somewhat natural to ask what if only one of those factors were normal. This is the idea behind semidirect products.

To motivate the construction of semidirect products, we first start by analyzing the case when H is a normal subgroup of G, and K is a subgroup of G. Recall that we have previously proven if H is a *normal* subgroup of G, and K is some subgroup of G, then HK is a subgroup of G. However, each element of HK is not so cleanly represented (see for example **??**). If we require that $H \cap K = 0$, then we get the additional property that every element of HK can be written *uniquely* as a product hk for some $h \in H, k \in K$. This means in particular that there is a bijection of sets $HK \to H \times K$.

At this point, we have collected a few ingredients to define the semidirect product of 2 groups. We have the underlying set: $H \times K$, and we would like H to be normal in our newly constructed group. We still need to figure out how the multiplication in our newly constructed group should work. This can be done by turning back to looking at how multiplication works in HK. If $h_1k_1, h_2k_2 \in HK$, then we wish to write their product $(h_1k_1)(h_2k_2)$ in the form hk for some $h \in H, k \in K$. The trick here is to make use of the normality of H by applying conjugation on h_2 by k_1. Indeed, we see that

$$\begin{aligned}(h_1k_1)(h_2k_2) &= h_1k_1h_2(k_1^{-1}k_1)k_2 \\ &= h_1(k_1h_2k_1^{-1})k_1k_2 \\ &= [h_1(k_1h_2k_1^{-1})][k_1k_2].\end{aligned}$$

We exploited the normality of H to ensure that $k_1h_2k_1^{-1} \in H$.

This gives us a sense on how to define the multiplication in our new semidirect product: given (h_1, k_1) and (h_2, k_2) in our underlying set, we define their product to be

$$(h_1(k_1h_2k_1^{-1}), k_1k_2).$$

But what on earth is $k_1h_2k_1^{-1}$? This doesn't make sense if H, K are some completely arbitrary groups. This is where another insight about conjugations come into play - the fact they are automorphisms. For a fixed k, we see that the map $\varphi_k : H \to H$ given by $h \mapsto khk^{-1}$ is an automorphism of H, since H is normal in G. Now, if we had some sort of map from K into the automorphisms of H, we can define what $k_1h_2k_1^{-1}$ meant. Let $\varphi : K \to \mathrm{Aut}(H)$ be such a map. We can then modify our product definition to be

$$(h_1\varphi(k_1)(h_2), k_1k_2),$$

which we can write in a cleaner way by defining $k_1 \cdot h_2$ to mean $\varphi(k_1)(h_2)$. We should check probably the most important quality: associativity. Does it work? Well,

$$((h_1, k_1)(h_2, k_2))(h_3, k_3) = (h_1(k_1 \cdot h_2), k_1k_2)(h_3, k_3) = (h_1(k_1 \cdot h_2)((k_1k_2) \cdot h_3), k_1k_2k_3),$$

and

$$(h_1, k_1)((h_2, k_2)(h_3, k_3)) = (h_1, k_1)(h_2(k_2 \cdot h_3), k_2k_3) = (h_1[k_1 \cdot (h_2(k_2 \cdot h_3))], k_1k_2k_3).$$

So associativity would hold if we had $h_1(k_2 \cdot h_2)((k_1k_2) \cdot h_3) = h_1[k_1 \cdot (h_2(k_2 \cdot h_3))]$. This is a bit messy, but not too bad. We can ignore the h_1 term (by applying the inverse of h_1 on the left), and expand $k_1 \cdot (h_2(k_2 \cdot h_3))$ to be $(k_1 \cdot h_2)k_1 \cdot (k_2 \cdot h_3)$, since $\varphi(k_1)$ is an automorphism. Thus we need

$$(k_1 \cdot h_2)(k_1 \cdot (k_2 \cdot h_3)) = (k_1 \cdot h_2)(k_1k_2) \cdot h_3.$$

Again, the $k_1 \cdot h_2$ term can be ignored, so what we really need is $k_1 \cdot (k_2 \cdot h_3) = (k_1k_2) \cdot h_3$. This would be true if φ is a *homomorphism*.

With that, we can finally give the

Definition 0.1 (Semidirect product)**.** Let H, K be groups, and let $\varphi : K \to \mathrm{Aut}(H)$ be a homomorphism. Then, the **semidirect product of** H **and** K, denoted $H \rtimes_\varphi K$, is the group with underlying set $H \times K$ and product

$$(h_1, k_2)(h_2, k_2) = (h_1 k_1 \cdot h_2, k_1 k_2).$$

We still have yet to check the existence of inverses and identities, but these are not hard, and we leave it to the reader as
Exercise 0.2. Verify that the construction of semidirect product above is actually a group.

Note that the notation $H \rtimes_\varphi K$ is chosen to remind us that H is normal in the semidirect product. If the homomorphism φ is clear we can drop the subscript and just write $H \rtimes K$. Let's discuss some properties of the construction we just created.

Proposition 0.3 (Properties of semidirect product)**.** Let H, K be groups, and let $\varphi : K \to \mathrm{Aut}(K)$ be a homomorphism. Let $H \rtimes K$ be the semidirect product of H and K, using φ. Then, the following hold:

(1) The order of $H \rtimes K$ is $|H||K|$.

(2) There are embeddings (injective homomorphisms) of H and K into $H \rtimes K$.

Moreover, if we identify H and K with their isomorphic copies, we have

(3) H is normal in G,

(4) $H \cap K = 0$,

(5) Conjugation of elements of h by elements of k is defined by $k \cdot h$, i.e. $khk^{-1} = k \cdot h$.

Proof. All of these follow very easily from the construction of the semidirect product, and are left to the reader in Exercise 0.4. □

Exercise 0.4. Prove Proposition 0.3

The following proposition is also often useful in showing that a semidirect product is the same as a direct product

Proposition 0.5. Let H, K be groups, φ a homomorphism of $K \to \mathrm{Aut}(H)$. Let $H \rtimes K$ denote their semidirect product. Then, the following are equivalent:

1. $H \rtimes K$ is isomorphic to the direct product $H \times K$ with the identity map,
2. φ is the trivial homomorphism,
3. K is normal in $H \rtimes K$.

Proof. Exercise; see Exercise 0.6 □

Exercise 0.6. Prove Proposition 0.5.

Now let's see some examples of semidirect products.
Example 0.7 (Dihedral groups)**.** Let H be a cyclic group of order n generated by r, and let K be a cyclic group of order 2, generated by s. Define $\varphi : K \to \mathrm{Aut}(H)$ by sending s to the inversion automorphism on H, which is $h \mapsto h^{-1}$. In the semidirect product, K has order 2 and H has order n. Notice that $shs^{-1} = h^{-1}$ for all $h \in H$. If we recall the presentation of D_n, we see these are precisely the relations that give D_n. Hence this semidirect product is actually D_n, so we conclude $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ is isomorphic to D_n. //

If we replace H with an infinite cyclic group, we get the infinite dihedral group. Of course, H can be any abelian group, since the inversion automorphism is valid provided H is abelian.