Now that we have the basic language to talk about rings, we shall discuss more examples of rings, which will come up time and time again.

0.1 Polynomial rings

The first class of rings we shall talk about are polynomial rings. These are extremely important in the study of rings.

The reader is likely already familiar with polynomials (as functions), which take on coefficients in the real numbers (or complex numbers). It is not much of a jump to allow polynomials to take on coefficients in an arbitrary ring. Let R be a commutative[1] ring with unity. Let x be an indeterminate. The *formal*[2] sum

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is called a **polynomial** in x with coefficients a_i in R. If a_n is nonzero then the **degree** of this polynomial is said to be n. The polynomial is said to be **monic** if $a_n = 1$. We define the set of all such polynomials to be the **polynomial ring in the indeterminate x with coefficients in R**, denoted $R[x]$.

The reader has probably already guessed how addition and multiplication should behave[3]. We simply add 2 polynomials by lining up their terms and adding their coefficients:

$$\begin{aligned}
&a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \\
+&b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0 \\
\hline
=&(a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_0 + b_0).
\end{aligned}$$

Here, some of the a_i's and b_j's could be zero, to make the equation work out. Of course, it is only reasonable to omit them when writing it down in a specific situation.

Multiplication is also as how one should expect. In this case we first define $(ax^i)(bx^j) = abx^{i+j}$, then we extend it in general to all polynomials by using distributivity, i.e.

$$\begin{aligned}
(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0) \cdot (b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0)& \\
&= a_n x^n (b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0) \\
&+ a_{n-1}x^{n-1}(b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0) \\
&+ \cdots \\
&+ a_0(b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0).
\end{aligned}$$

In general, the coefficient of x^k in the product is given by $\sum_{i=0}^{k} a_i b_{k-i}$[4].

With these operations, it is not too difficult to verify $R[x]$ forms a ring. Moreover, there is an isomorphic copy of R embedded in $R[x]$, namely, the constant polynomials.

When R is an integral domain, the following familiar properties of polynomials with integer coefficients hold:

Proposition 0.1. Let R be an integral domain, and $p(x), q(x) \in R[x]$ be nonzero polynomials. Then,

1. $\deg p(x)q(x) = \deg p(x) + \deg q(x)$,
2. if $u(x) \in R[x]$ is a unit then it is a constant polynomial,
3. $R[x]$ is an integral domain as well.

[1] It is possible to define polynomial rings over noncommutative rings, but we leave this to the exercises since they're not used much.
[2] For the sake of intuition and exposition we will not define this in terms of sequences. We leave this definition to the exercises.
[3] This is why we did not choose to define these with sequences
[4] Some readers may recognize this as convolution.

Proof. The proof is routine. □

Exercise 0.2. Prove the previous proposition.