Group presentations are a tool for us to describe all the elements of a group. We have already made use of them to talk about the dihedral group. We shall only give a light overview here; they will be treated more formally later on.

**Definition 0.1** (Generator). Let $G$ be a group and let $S \subseteq G$. Then if every $g \in G$ has the property that $g$ can be written as the finite product of elements of $S$ and their inverses, then $S$ is called a set of **generators for** $G$. We thus say that $G$ is *generated by* $S$.

We leave it to the exercises to formalize this notion. For now, an intuitive understanding will suffice. Let us now discuss notation. If $S$ is a set of generators for $G$, we shall write $G = \langle S \rangle$. If $S$ is a finite set, say $S = \{ g_1, \ldots, g_n \}$, then we shall write $G = \langle g_1, \ldots, g_n \rangle$ instead.

**Definition 0.2** (Relation). Let $G$ be a group and suppose $S$ generates $G$. Any equation that generators satisfy is a called a **relation**.

**Example 0.3** (Presentation of $\mathbb{Z}$). The reader has probably already guessed this. Every element of $\mathbb{Z}$ is of the form $1 + \cdots + 1$ where you add 1 to itself $n$ times to obtain $n$. It thus follows that $\mathbb{Z} = \langle 1 \rangle$. We also notice that we can actually write any element as $-(-1 + \cdots + -1)$, adding $-1$ to itself $n$ times and taking the inverse of it. Thus $\mathbb{Z} = \langle -1 \rangle$ too. It's not too hard to see that any other element of $\mathbb{Z}$ cannot be a generator of $\mathbb{Z}$. //

Our main focus here shall be on the presentation of $D_n$. Before we can find ourselves a presentation for $D_n$, we must first take a look at some of the properties of $D_n$. Consider a regular $n$-gon, and let $r$ be a rotation of $360/n$ degrees counterclockwise. Let $s$ be reflection across the line between the vertex 1 and the origin. For a helpful visual, see Figure 1.

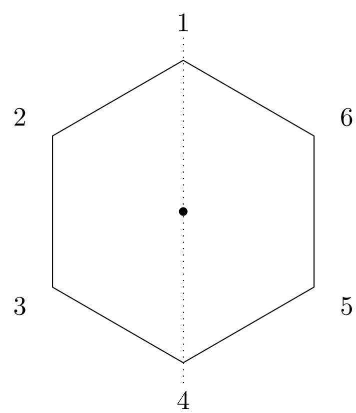


Figure 1: Labelled hexagon

Now, the following details can be easily deduced. We leave the details to the reader in Exercise 0.8.

1. The order of $r$ is $n$. This says that every rotation is distinct.
2. The order of $s$ is 2. This says that applying the reflection twice leaves the $n$-gon unchanged.
3. For any $i$, $s \neq r^i$. This says that a rotation is never a reflection.
4. Whenever $i \neq j$, $sr^i \neq sr^j$ for $i, j \in \{ 0, \ldots, n-1 \}$. As such,

$$D_n = \{ e, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1} \}.$$

   This means every element of $D_n$ can be written *uniquely* in the form $sr^k$ for some $k \in \{ 0, \ldots, n-1 \}$.

5. $r^j s = sr^{-j}$ for $j \in \{ 0, \ldots, n-1 \}$. This is better understood by seeing that $rs = sr^{-1}$. The reader is encouraged to pull out something that's square (or rectangular) and try this for themselves.

With these facts, we are now ready to construct a presentation of $D_n$. From 4, every element of $D_n$ can be written with $r$ and $s$, so we would have 2 generators: $r, s$. At this point, we have no relations yet, but it seems sensible that we should write down the relations $r^n = e$ and $s^2 = e$. For our last relation, we shall write down $r^j s = sr^{n-j}$, a

slight modification of number 5. Our choice for this relation is forced by the fact that the other facts simply say that something is not equal to something else. We now present[1] to the reader, the presentation of $D_n$.
**Example 0.4** (Presentation of $D_n$)**.** The usual presentation of $D_n$ is given by

$$D_n = \langle r, s \mid r^n = s^2 = e,\ sr^j = r^{-j}s \rangle.$$

Intuitively, $r$ is a rotation and $s$ is a reflection. We leave it to the reader to check that this presentation actually gives us $D_n$.

Of course, there are other presentations, such as

$$D_n = \langle a, b \mid a^2 = b^2 = (ab)^n = e \rangle.$$

You can think about it as $a = s$ and $b = sr$ where $s, r$ are from the first presentation. //

Group presentations are nice because they're a compact way to describe a group. Unfortunately, there are some caveats to group presentations. Due to the flexibility of group presentations, we do not require that the generators come from some preexisting group. What this means is that we can write down some presentation like $\langle a, b \mid a^4 = b^2 = e \rangle$ and consider all the strings formed by $a$ and $b$ and their formal inverses[2]. What this means is that this presentation defines a group $G$ where the set is all finite strings with letters $a, b$ and letters $a^{-1}, b^{-1}$, with the property that $aa^{-1}$, $a^{-1}a$ and $bb^{-1}, b^{-1}b$ are removed from the string. For example, the string $aab^{-1}ba$ is equal to $aaa$. The same conventions apply: if we have $aaaa \cdots a$ $n$ times, we would write $a^n$ instead. Such a construction is called a *free group*. The relations then specify what strings are equal in this group. We will return to the concept of free groups in a latter chapter, but because of this, if we are given an arbitrary presentation, it can be difficult or impossible to distinguish between distinct elements. In the example with $D_n$, we worked backwards by deducing facts that the generators must satisfy and property 4 told us that everything in $D_n$ was able to be uniquely expressed in terms of the generators and relations, but this may not be true for an arbitrary group presentation. This has some nasty consequences.
**Example 0.5** (A group presentation that leads to an infinite group)**.** Consider the presentations

$$\langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle \tag{1}$$

$$\langle a, b \mid a^3 = b^3 = (ab)^3 = e \rangle \tag{2}$$

What do you think the order of Equation (1) is? 2? 4? It turns out that this is a group of order 4. (Actually this turns out to be $D_2$. See Exercise 0.10) Now what about Equation (2)? Is it 3? 9? No! It's an infinite group. As such, one must not get misled by things like

$$\langle x, y, z \mid x^n = y^k = z^m = e, \cdots \rangle$$

and conclude that the group is necessarily finite. //

Another important remark is in order. Given a group presentation, we cannot assume that the relations as written are the only relations. That is, there may be some hidden relations.
**Example 0.6.** This is taken from [DF04, Eqn 1.2, p. 26]. Let

$$X_n = \langle x, y \mid x^n = y^2 = 1,\ xy = yx^2 \rangle.$$

Although $X_n$ looks like a group that has order $2n$. This is not true. The problematic relationship is $xy = yx^2$. Let's now see why this causes problems. First, notice that $y$ has order 2, so that $y^2 = e$. Now we consider the relationship $x = xy^2$. Now, $y^2 = yy$, so then we have

$$x = (xy)y = (yx^2)y = (yx)(xy) = (yx)(yx^2) = y(xy)x^2 = y(yx^2)(x^2) = x^4.$$

So this tells us that $x^3 = e$. So the order of $X_n$ can be at most 6. //
**Example 0.7** (A group with an elaborate presentation that degenerates)**.** This example is from [DF04, Eqn 1.3, p. 27]. Let

$$Y = \langle u, v \mid u^4 = v^3 = 1,\ uv = v^2u^2 \rangle.$$

While the first relation may suggest that $Y$ has order 12, $Y$ turns out to actually be the trivial group. A sketch of this proof is given in Exercise 0.11. //

[1] lmao
[2] This is a horrible name and very pedagogically disastrous, I'll need to change this soon

Now why does this not happen with the presentation we gave for $D_n$? The reason is because we crafted a presentation from properties that the group already satisfies. As such, we have demonstrated that there is a group with generators $r, s$ that satisfy the relations as given in the standard presentation. This tells us that a group which satisfies the relations of the standard presentation of $D_n$ would have at least order $2n$, since it would contain $D_n$. It can also be proven that any group with the presentation as given would have order at most $2n$, so necessarily this presentation gives us the dihedral group.

### 0.0.1 Problems and Exercises

**Exercise 0.8** (Properties of $D_n$)**.** Prove the following properties about $D_n$.

1. The order of $r$ is $n$.
2. The order of $s$ is 2.
3. For any $i$, $s \neq r^i$.
4. Whenever $i \neq j$, $sr^i \neq sr^j$ for $i, j \in \{0, \ldots, n-1\}$.
5. $r^j s = sr^{-j}$ for $j \in \{0, \ldots, n-1\}$. A good strategy is to prove that $rs = sr^{-1}$ first, then apply induction on $j$.

**Exercise 0.9.** Find a presentation of $\mathbb{Z}_n$.

**Exercise 0.10.** Show that the presentation in Equation (1) gives the dihedral group $D_2$, but that the presentation in Equation (2) is a presentation of an infinite group.

**Exercise 0.11.** We shall prove that $Y$ as defined in Example 0.7 is the trivial group.

1. Show that $v^2 = v^{-1}$.
2. Prove that $v^{-1}u^3v = u^3$. To get started, notice that $v^{-1} = v^2$, and so $v^2u^3v = (v^2u^2)(uv)$. You will need to make use of part 1 again.
3. Prove that $u^3$ and $v$ commute.
4. Prove that $Y$ is abelian. Note that it suffices to show that $u$ and $v$ commute (why?). Try to prove that $u^9 = u$, and then apply (2)
5. Prove that $uv = e$, $u = e$ and $v = e$. Conclude that $Y$ is the trivial group.

**Problem 0.1.** Let $G$ be a finitely generated group, and suppose that $[G : H]$ is finite. Prove that $H$ is finitely generated. *(You might need the content from* **??** *to do this.)*

*Bonus: Try to find a proof of this fact using algebraic topology*